A General Impossibility Result on Strategy-Proof Social Choice Hyperfunctions *

Selçuk Özyurt †
and
M. Remzi Sanver ‡

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Abstract

A *social choice hyperfunction* picks a non-empty set of alternatives at each admissible preference profile over sets of alternatives. We analyze the manipulability of social choice hyperfunctions. We identify a domain D^λ of lexicographic orderings which exhibits an impossibility of the Gibbard-Satterthwaite type. Moreover, this impossibility is inherited by all well-known superdomains of D^λ. As most of the standard extension axioms induce superdomains of D^λ while social choice correspondences are particular social choice hyperfunctions, we are able to generalize many impossibility results in the literature.

Keywords: Strategy-proofness, Manipulation, Gibbard-Satterthwaite Theorem, Social Choice Correspondences, Hyperfunctions

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†Department of Economics, New York University, NY, 10012 USA

‡Department of Economics, Istanbul Bilgi University, 34387, Kustepe, Istanbul, Turkey

1 Introduction

The seminal impossibility result of Gibbard (1973) and Satterthwaite (1975) shows that every non-manipulable social choice function which is defined over the unrestricted domain of preference relations is dictatorial if its range contains at least three alternatives. This result is fairly robust. There is literature dating back to Pattanaik (1973), Barberà (1977), Kelly (1977), Gärdenfors (1978) and Feldman (1979a, 1979b, 1980), showing similar impossibilities for social choice rules which are not necessarily singleton-valued. Their results are given in a framework where social choice rules are modeled as social choice correspondences which assign a set of alternatives to every preference profile over alternatives. The analysis is made under certain extension axioms which connect preferences over alternatives to preferences over sets of alternatives.

There is a recent trend of carrying this analysis further in a framework where manipulability is analyzed via *hyperfunctions*, i.e. functions that pick a non-empty set of alternatives at each admissible preference profile over sets of alternatives. This approach has the advantage of being part of a more general framework which, when compared to the classical one, allows to use finer information about individual preferences over sets. Standard social choice correspondences impose a strong invariance condition over social choice rules: A social outcome must be the same as long as individuals' rankings over singleton sets are the same. In other words, even if individuals change their preferences over sets, the social outcome must remain unchanged as long as their ordering of singleton sets remains the same. It is clear that hyperfunctions do not have such a restriction and thus, are more general social choice objects. As a result, every Gibbard-Satterthwaite type of impossibility established in the world of hyperfunctions can be carried to the standard world of social choice correspondences.

Of course, if we consider social choice hyperfunctions defined over the full domain of preference profiles, then strategy-proofness is equivalent to dictatorship, simply by the Gibbard-Satterthwaite theorem. Barberà, Dutta and Sen (2001) -BDS from now on- show that this equivalence is quite robust under domain restrictions. They consider a domain of orderings over sets defined through the idea of expected utility consistency. They use two versions of this concept. One version leads to a coarser domain, ending up in an impossibility of the Gibbard-Satterthwaite type: Any unanimous and strategy-proof social choice hyperfunction must be dictatorial. The other version, leading to a narrower domain, allows for a slightly more permissive result, though still of the Gibbard-Satterthwaite spirit: Any unanimous and strategy-proof social choice hyperfunction must be either dictatorial or bi-dictatorial.

We identify a lexicographic domain D^λ of orderings over sets which exhibits a similar impossibility: Every unanimous and strategy-proof social choice hyperfunction defined over D^λ is dictatorial or bi-dictatorial. In fact, D^λ extracts the essential structure of the BDS domains leading to the impossibility of the Gibbard-Satterthwaite type. We are thus able to establish that the impossibility over D^λ is inherited by all of its well-known superdomains. As most of the standard extension axioms induce superdomains of D^λ, we are able to state our impossibility result for many well-known domains, including those determined by the extension axioms that Gärdenfors (1976) and Kelly (1977) use in their analysis of strategy-proof social choice correspondences.

Section 2 introduces the preliminaries. Section 3 identifies the lexicographic domain D^λ over which an impossibility result of the Gibbard-Satterthwaite type prevails. Section 4 announces the main result which extends this impossibility to the superdomains of D^λ. Section 5 gives instances of impossibilities generalized by our main result. Section 6 makes some closing remarks and discusses related literature.

2 Preliminaries

Taking any two integers $n \geq 2$ and $m \geq 3$, we consider a society $\mathbf{N} = \{1, \cdots, n\}$ confronting a set of alternatives $\mathbf{A}$ with m alternatives. We write $\underline{\mathbf{A}} = 2^{\mathbf{A}} \diagdown \{\emptyset\}$ for the set of non-empty subsets of $\mathbf{A}$.

We let Π stand for the set of complete, transitive and antisymmetric binary relations over $\mathbf{A}$. Every $\rho \in \Pi$ represents an individual preference on the elements of $\mathbf{A}$ in the following manner: For any $a, b \in \mathbf{A}$, $a \ \rho \ b$ means "a is at least as good as b".[1] When the preference over $\mathbf{A}$ is specified to belong to a particular agent $i \in \mathbf{N}$, we write it as ρ_i. A typical preference profile over $\mathbf{A}$ is denoted by $\underline{\rho} = (\rho_1, \cdots, \rho_n) \in \Pi^{\mathbf{N}}$.

Similarly, we let $\Re$ stand for the set of all complete and transitive binary relations over $\underline{\mathbf{A}}$. Every $R \in \Re$ represents an individual preference on the elements of $\underline{\mathbf{A}}$ in the

[1] As ρ is antisymmetric, for any distinct $a, b \in \mathbf{A}$ we have $a \ \rho \ b \Rightarrow$ not $b \ \rho \ a$. In other words, for distinct alternatives $a \ \rho \ b$ means "a is preferred to b"

following manner: For any $X, Y \in \underline{\mathbf{A}}$, $X\ R\ Y$ means "X is at least as good as Y". We denote P and I for the strict and indifference counterparts of R.[2] In case the preference over $\underline{\mathbf{A}}$ is specified to belong to a particular agent $i \in \mathbf{N}$, we write it as R_i, with its respective strict and indifference counterparts P_i and I_i. A typical preference profile over $\underline{\mathbf{A}}$ is denoted by $\underline{R} = (R_1, \cdots, R_n) \in \Re^{\mathbf{N}}$.

Given any $D \subseteq \Re$, we define a *social choice hyperfunction* (or simply a "hyperfunction") as a mapping $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$. We only consider hyperfunctions whose domains are Cartesian products of some non-empty $D \subseteq \Re$, in which case we say that the hyperfunction is defined over the domain D. A hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ is *manipulable over* E ($\emptyset \neq E \subseteq D$) if and only if there exists $i \in \mathbf{N}$ and $\underline{R}, \underline{R}' \in E^{\mathbf{N}}$ with $R_j = R'_j$ for all $j \in \mathbf{N} \setminus \{i\}$ such that $f(\underline{R}')\ P_i\ f(\underline{R})$, and *strategy-proof over* E if f is not manipulable over E. Finally, f is *strategy-proof* (resp. *manipulable*) if it is strategy-proof over D (resp. manipulable over D).

A hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ is *dictatorial* if and only if there exists $d \in \mathbf{N}$ such that for all $\underline{R} \in D^{\mathbf{N}}$ we have $f(\underline{R}) \in \arg\max_{\underline{\mathbf{A}}} R_d$, *bi-dictatorial* if there exist distinct $i, j \in \mathbf{N}$ such that for all $\underline{R} \in D^{\mathbf{N}}$ we have $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_i \cup \arg\max_{\underline{\mathbf{A}}} R_j$ and *unanimous* if for any $X \in \underline{\mathbf{A}}$ and any $\underline{R} \in [D]^{\mathbf{N}}$ with $\arg\max_{\underline{\mathbf{A}}} R_i = X$ for all $i \in \mathbf{N}$, we have $f(\underline{R}) = X$. A domain D is *dictatorial* (resp. *bi-dictatorial*) if and only if every unanimous and non-manipulable hyperfunction defined over D is dictatorial (resp. dictatorial or bi-dictatorial).

Given any $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$, $E \subseteq D$ and distinct $i, j \in \mathbf{N}$, we say that i and j are the *bi-dictators* of f over E if and only if $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_i \cup \arg\max_{\underline{\mathbf{A}}} R_j$ for all $\underline{R} \in [E]^{\mathbf{N}}$, and i is the *dictator* of f over E if $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_i$ for all $\underline{R} \in [E]^{\mathbf{N}}$.

An immediate consequence of the Gibbard-Satterthwaite theorem is that $\Re$ is dictatorial. In this paper, we ask whether it is possible to escape this impossibility by restricting $\Re$ through axioms which extend preferences over alternatives to sets of alternatives.

We accept that if the preference over $\mathbf{A}$ is some $\rho \in \Pi$, then the preference over $\underline{\mathbf{A}}$ can be some $R \in \Re$ which is "consistent" with ρ. Thus, we define a consistency map

[2] For any $X, Y \in \underline{\mathbf{A}}$, we write $X\ P\ Y$ if and only if $X\ R\ Y$ holds but $Y\ R\ X$ does not, i.e. X is preferred to Y. In case $X\ R\ Y$ and $Y\ R\ X$ both hold, we write $X\ I\ Y$, which means indifference between X and Y.

$\kappa : \Pi \to 2^{\Re} \diagdown \{\emptyset\}$ which assigns to every $\rho \in \Pi$ a non-empty set $\kappa(\rho) \subseteq \Re$ of preferences on $\underline{\mathbf{A}}$ consistent with ρ. We assume that every consistency map κ satisfies the following basic axiom **A0**:

A0. Given any $\rho \in \Pi$ and any $R \in \kappa(\rho)$, we have $x \ \rho \ y \Leftrightarrow \{x\} \ R \ \{y\}$ for all $x, y \in \mathbf{A}$.

A0 requires that the ordering of individuals over singleton sets must be the same as their ordering over the basic alternatives. Remark that **A0** implies $\kappa(\rho) \cap \kappa(\rho') = \emptyset$ for all distinct $\rho, \rho' \in \Pi$.

Given any consistency map κ, we write $D^\kappa = \bigcup_{\rho \in \Pi} \kappa(\rho)$ for the set of *acceptable* preferences over $\underline{\mathbf{A}}$ induced by κ. Note that D^κ is always a strict subset of $\Re$, as every κ is assumed to satisfy our basic axiom **A0**.

Now, we introduce the concept of a leximax extension over $\underline{\mathbf{A}}$ which orders any two sets according to their best elements. If these are the same, then the ordering is made according to the second best elements, and so on. The elements according to which the sets are compared will disagree at some step - except possibly when one set is a subset of the other, in which case the smaller set is preferred.[3] Formally, given an ordering $\rho \in \Pi$, the leximax ordering $\lambda^+(\rho)$ over $\underline{\mathbf{A}}$ is defined as follows: Take any distinct $X, Y \in \underline{\mathbf{A}}$. Let, without loss of generality, $X = \{x_1, \cdots, x_{|X|}\}$ and $Y = \{y_1, \cdots, y_{|Y|}\}$ such that $x_j \ \rho \ x_{j+1}$ for all $j \in \{1, \cdots, |X|-1\}$ and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1, \cdots, |Y|-1\}$. If $x_h = y_h$ for all $h \in \{1, \cdots, \min\{|X|, |Y|\}\}$, then $X \ \lambda^+(\rho) \ Y$ if and only if $|X| < |Y|$. If there exists some $h \in \{1, \cdots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$, then $X \ \lambda^+(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest value of h.

The leximin extension over $\underline{\mathbf{A}}$ orders any two sets according to a lexicographic comparison of their worst elements - this is the mirror image of the leximax extension.[4] So given an ordering $\rho \in \Pi$, the leximin ordering $\lambda^-(\rho)$ over $\underline{\mathbf{A}}$ is defined as follows: Take any distinct $X, Y \in \underline{\mathbf{A}}$. Let, without loss of generality, $X = \{x_1, \cdots, x_{|X|}\}$ and $Y = \{y_1, \cdots, y_{|Y|}\}$ such that $x_{j+1} \ \rho \ x_j$ for all $j \in \{1, \cdots, |X|-1\}$ and $y_{j+1} \ \rho \ y_j$ for all $j \in \{1, \cdots, |Y|-1\}$. If $x_h = y_h$ for all $h \in \{1, \cdots, \min\{|X|, |Y|\}\}$, then $X \ \lambda^-(\rho) \ Y$ if and only if $|X| > |Y|$. If there exists some $h \in \{1, \cdots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$, then $X \ \lambda^-(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest value of h.

[3] This is exactly how words are ordered in a dictionary. For example, given three alternatives a, b and c, the leximax extension of the ordering $a \ b \ c$ is $\{a\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \{b\}, \{b, c\}, \{c\}$.

[4] For example, the leximin extension of the ordering $a \ b \ c$ is $\{a\}, \{a, b\}, \{b\}, \{a, c\}, \{a, b, c\}, \{b, c\}, \{c\}$.

Kaymak and Sanver (2003) show that at each $\rho \in \Pi$, the leximax and leximin extensions respectively determine unique orderings $\lambda^{+}(\rho)$ and $\lambda^{-}(\rho)$ over $\underline{\mathbf{A}}$ which are complete, transitive and antisymmetric.

We write λ for the consistency map which assigns the leximax and leximin extensions to every $\rho \in \Pi$, i.e. $\lambda(\rho) = \{\lambda^{+}(\rho), \lambda^{-}(\rho)\}$ for all $\rho \in \Pi$. We write $D^{\lambda} = \bigcup_{\rho\in\Pi} \lambda(\rho)$ for the set of acceptable preferences over $\underline{\mathbf{A}}$ defined through λ.

3 A Bi-dictatorial Domain

In this section, we show that D^{λ} is bi-dictatorial. Notice first that under the consistency map λ, the best and the worst elements of every ordering is a singleton set. This is what we call the "regularity" of a domain. We say that a domain D is *regular* if D consists of orderings having singleton sets as their unique maximal and minimal elements. Regularity is very natural when we conceive sets as non-resolute outcomes. A regular domain D is called *fully regular* if every singleton set is a unique maximal and a unique minimal element for at least one ordering in D. So D^{λ} is fully regular. Note that the range of every unanimous hyperfunction defined over a fully regular domain contains all singleton sets.

Theorem 3.1. A unanimous hyperfunction $f : [D^{\lambda}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial or bi-dictatorial.

We prove Theorem 3.1 in Appendix A.

Remark 3.1. Theorem 3.1 no longer holds when the domain of the hyperfunction is further restricted to $D^{\lambda^{+}} = \bigcup_{\rho\in\Pi}\{\lambda^{+}(\rho)\}$ or to $D^{\lambda^{-}} = \bigcup_{\rho\in\Pi}\{\lambda^{-}(\rho)\}$. To see the former, consider the hyperfunction $f_1 : [D^{\lambda^{+}}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ that picks a strict majority winner when it exists and the whole set $\mathbf{A}$ otherwise.[5] So, for every $\underline{R} \in [D^{\lambda^{+}}]^{\mathbf{N}}$ we have

$$f_1(\underline{R}) = \begin{cases} \{a\} & \text{if } \left|\{i \in \mathbf{N} : \arg\max_{\underline{\mathbf{A}}} R_i = \{a\}\}\right| > n/2 \text{ for some } a \in \mathbf{A} \\ \mathbf{A} & \text{otherwise} \end{cases}$$

One can check that f_1 is unanimous, neither dictatorial nor bi-dictatorial. It is non-manipulable over $D^{\lambda^{+}}$ because, at any preference profile where the outcome is some

[5]This social choice rule is given as Example 5 in Benoit (2002) with a similar purpose.

singleton, there is a majority of voters obtaining their best alternative (thus having no incentive to manipulate) while the remaining voters are unable to manipulate. On the other hand, at a preference profile where the outcome is the set $\mathbf{A}$, all voters prefer $\mathbf{A}$ to all singletons -except their first best since they have leximax preferences.

To see that Theorem 3.1 fails to hold over D^{λ^-}, consider the hyperfunction $f_2 : [D^{\lambda^-}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ which picks the union of the top alternatives of all agents.[6] So, for every $\underline{R} \in [D^{\lambda^-}]^{\mathbf{N}}$ we have $f_2(\underline{R}) = \bigcup_{i\in\mathbf{N}} \arg\max_{\underline{\mathbf{A}}} R_i$. Again one can check that f_2 is unanimous, neither dictatorial nor bi-dictatorial. On the other hand, by a unilateral misrepresentation, a voter can possibly make an alternative, which is not her best, included in the outcome set with the possible expense of making her first best excluded. Though whether she is able to include (or exclude) an alternative depends on the preference profile, misrepresentation is never beneficial under leximin preferences. Thus, f_2 is non-manipulable over D^{λ^-}.[7]

Remark 3.2. Theorem 3.1 fails to hold when $|\mathbf{A}| = 2$. Note that even when $|\mathbf{A}| = 2$, $|\underline{\mathbf{A}}| = 3$. So, the range condition of the Gibbard-Satterthwaite theorem holds. However, when $|\mathbf{A}| = 2$, both f_1 and f_2 defined over D^{λ} are strategy-proof.

[6] This social choice rule is given as Example 6 in Benoit (2002), aiming to show that the results of BDS do not apply to domains obtained through lexicographic orderings.

[7] The set of non-manipulable social choice correspondences under leximin preferences is characterized by Campbell and Kelly (2002) who, among other things, show that a social choice correspondence which is strategy-proof over a domain that contains the leximin extensions must be choosing, at each preference profile, the union of the top elements of a fixed coalition H of voters. In case this result is carried a step forward so as to show that over D^{λ} this coalition H can be at most a doubleton, we get the restriction of Theorem 3.1 to social choice correspondences.

4 Main Result

In this section, we analyze the implications of Theorem 3.1 for the superdomains of D^λ. Although one may be tempted to think that all superdomains of D^λ are dictatorial or bi-dictatorial, this is not the case, as the following example illustrates:[8]

Consider a set of alternatives $\mathbf{A} = \{a, b, c\}$. Now take the ordering $R \in \Re$ over $\underline{\mathbf{A}}$ where $\arg\max_{\underline{\mathbf{A}}} R = \{a, b\}$ and $\{a, b\}\ P\ \{a\}\ P\ X$ for all $X \in \underline{\mathbf{A}} \diagdown \{\{a, b\}, \{a\}\}$. The following hyperfunction defined over the domain $D^\lambda \cup \{R\}$ is unanimous, strategy-proof, neither dictatorial nor bi-dictatorial:
For all $\underline{R} \in [D^\lambda \cup \{R\}]^{\mathbf{N}}$, we have

$$f(\underline{R}) = \begin{cases} \arg\max_{\underline{\mathbf{A}}} R_1 & \text{if } R_1 \in D^\lambda \\ \arg\max_{\{\{a,b\},\{a\}\}} R_2 & \text{otherwise} \end{cases}$$

Thus, not every superdomain of D^λ is dictatorial or bi-dictatorial. So, it is of interest to see which superdomains of D^λ preserve the property of being dictatorial or bi-dictatorial.

Proposition 4.1. Take two fully regular domains $D, D' \subset \Re$ with $D' \subset D$, and a strategy-proof hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$. If $d \in \mathbf{N}$ is the dictator of f over D' then d is the dictator of f over D.

Proof. Take D, D' and f as in the statement of the proposition. Let, without loss of generality, $1 \in \mathbf{N}$ be the dictator over D'. Take any $\underline{R} \in D^{\mathbf{N}}$ with $R_1 \in D'$. Write $k = |\{i \in \mathbf{N} \diagdown \{1\} : R_i \in D \diagdown D'\}|$. We claim that $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1$ for any value of k in $\{1, \cdots, n-1\}$. We prove our claim by induction on k. We present the proof only for $k = 1$ while these arguments, mutatis mutandis, establish the inductive step. So take $k = 1$ and let, without loss of generality, $R_2 \in D \diagdown D'$. Suppose $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \diagdown \{\{a\}\}$ where $\{a\} = \arg\max_{\underline{\mathbf{A}}} R_1$. Now, take some $R'_2 \in D'$ with $\arg\min_{\underline{\mathbf{A}}} R'_2 = \{a\}$. Consider the profile $\underline{R}' \in [D']^{\mathbf{N}}$ where $R'_j = R_j$ for all $j \in \mathbf{N} \diagdown \{2\}$ while the preference of agent 2 is R'_2. As agent 1 is the dictator over D', we have $f(\underline{R}') = \{a\}$. But since $X\ P'_2\ \{a\}$, agent 2 manipulates $\underline{R}'$ by $R_2 \in D \diagdown D'$, contradicting that f is strategy-proof, and this proves our claim for $k = 1$.

[8]Sanver (2007) characterizes dictatorial domains which do not admit non-dictatorial superdomains.

Note that for $k = 0$, we have $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1$ by the choice of D' and f. Thus, $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1$ for all $\underline{R} \in D^{\mathbf{N}}$ with $R_1 \in D'$, which implies $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1$ for all $\underline{R} \in D^{\mathbf{N}}$ with $R_1 \in D$ as well, as otherwise 1 manipulates $\underline{R}$ by picking a preference in D'. $\square$

Before stating the next proposition, we introduce a few conditions. The first one, which we refer to as Condition β, is used by Barberà (1977) in his analysis of strategy-proof social choice correspondences. We define two versions of it. We say that a fully regular domain $D \subseteq \Re$ satisfies condition β^* (resp. β) if and only if for any $R \in D$ with $\arg\max_{\underline{\mathbf{A}}} R = \{a\}$ we have

(β^*) $\quad \{a\}\ P\ \{a,b\}\ P\ \{b\}$ for all $b \in \mathbf{A} \setminus \{a\}$.

(β) $\quad \{a\}\ P\ \{a,b\}\ R\ \{b\}$ for all $b \in \mathbf{A} \setminus \{a\}$.

Clearly β^* implies β.

The second condition which imposes a weak form of separability on preferences over sets has also two versions. We say that a fully regular domain $D \subseteq \Re$ satisfies condition σ^* (resp. σ) if and only if for any $R \in D$ with $\arg\max_{\underline{\mathbf{A}}} R = \{a\}$ we have

(σ^*) $\quad \{a,b\}\ P\ \{c,b\}$ for all $b, c \in \mathbf{A} \setminus \{a\}$.

(σ) $\quad \{a,b\}\ R\ \{c,b\}$ for all $b, c \in \mathbf{A} \setminus \{a\}$.

Proposition 4.2. Let $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ be a strategy-proof hyperfunction with $D \supset D^{\lambda}$ being a (fully regular) domain satisfying conditions β^* and σ^*. If $i, j \in \mathbf{N}$ are the bi-dictators of f over D^{λ} then i and j are the bi-dictators of f over D.

The proof of Proposition 4.2 is given in Appendix B.

Interestingly, conditions β and σ turn out to be necessary and sufficient to ensure the strategy-proofness of bi-dictatorial hyperfunctions. We state this in the following proposition.

Proposition 4.3. Take any fully regular domain $D \supset D^{\lambda}$. A bi-dictatorial hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ is strategy-proof if and only if D satisfies conditions β and σ.

Proof. We leave the "if" part to the reader. To show the "only if" part, consider any fully regular domain $D \supset D^{\lambda}$ and a bi-dictatorial hyperfunction f defined over D. Let, without loss of generality, individuals 1 and 2 be the bi-dictators. We show that f is manipulable if β or σ fails to hold. First suppose β is violated. Let $R \in D$ with $\arg\max_{\underline{\mathbf{A}}} R = \{a\}$ be an ordering which violates β. Hence $\{b\}\ P\ \{a, b\}$ for some $b \in \mathbf{A} \diagdown \{a\}$. Consider a preference profile $\underline{R} \in D^{\mathbf{N}}$ with $R_1 = R$ and $\arg\max_{\underline{\mathbf{A}}} R_2 = \{b\}$. As f is bi-dictatorial, we have $f(\underline{R}) = \{a, b\}$. Thus, agent 1 can manipulate f at $\underline{R}$ by pretending any ordering R_1' with $\arg\max_{\underline{\mathbf{A}}} R_1' = \{b\}$. Now suppose σ is violated. Let $R \in D$ with $\arg\max_{\underline{\mathbf{A}}} R = \{a\}$ be an ordering which violates σ. Hence we have $\{b, c\}\ P\ \{a, b\}$ for some $b, c \in \mathbf{A} \diagdown \{a\}$. Consider a preference profile $\underline{R} \in D^{N}$ with $R_1 = R$ and $\arg\max_{\underline{\mathbf{A}}} R_2 = \{b\}$. As f is bi-dictatorial, we have $f(\underline{R}) = \{a, b\}$. But since $\{b, c\}\ P_1\ \{a, b\}$, agent 1 can manipulate f at $\underline{R}$ by pretending an ordering R_1' with $\arg\max_{\underline{\mathbf{A}}} R_1' = \{c\}$. $\square$

Propositions 4.1, 4.2 and 4.3 lead to the following theorem which extends Theorem 3.1 to almost all superdomains of D^{λ}.

Theorem 4.1. Take any fully regular domain $D \supset D^{\lambda}$ satisfying condition β^* and consider a unanimous hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$.

(i) If D satisfies condition σ^*, then f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

(ii) If D violates condition σ, then f is strategy-proof if and only if f is dictatorial.

The proof of Theorem 4.1 is given in Appendix C.

Remark 4.1. D^{λ} itself satisfies conditions β^* and σ^*, hence Theorem 3.1 follows from Theorem 4.1.

Remark 4.2. Condition β^* is critical for Theorem 4.1 to hold. To see this, let $\mathbf{A} = \{a, b, c\}$ and $\mathbf{N} = \{1, 2\}$. Consider $D = D^{\lambda} \cup \{R^*\} \subset \Re$ where $\arg\max_{\underline{\mathbf{A}}} R^* = \{a\}$, $\arg\min_{\underline{\mathbf{A}}} R^* = \{b\}$, $\{c\}\ P^*\ \{a, c\}$, $\{c\}\ P^*\ \{b, c\}$, $\{a, b\}\ I^*\ \{b, c\}$ and $\{a, b\}\ P^*\ \{b\}$. As $\{a\}\ P^*\ \{c\}\ P^*\ \{a, c\}$, the domain D violates Condition β^*. One can check that hyperfunction f defined below is unanimous, strategy-proof, non-dictatorial and non bi-dictatorial: For any $R_1, R_2 \in D$ we have

$$f(R_1,R_2)=\begin{cases}\{a,b\} & \text{if } R_i=R^*, \arg\max_{\underline{\mathbf{A}}} R_j=\{b\} \text{ and } \arg\max_{\underline{\mathbf{A}}\setminus\{b\}} R_j=\{a,b\}\\ \{b,c\} & \text{if } R_i=R^*, \arg\max_{\underline{\mathbf{A}}} R_j=\{b\} \text{ and } \arg\max_{\underline{\mathbf{A}}\setminus\{b\}} R_j=\{b,c\}\\ \{c\} & \text{if } R_i=R^* \text{ and } \arg\max_{\underline{\mathbf{A}}} R_j=\{c\}\\ X & \text{otherwise}\end{cases}$$

where $i,j\in\{1,2\}, i\neq j$ and $X=\arg\max_{\underline{\mathbf{A}}} R_1 \bigcup \arg\max_{\underline{\mathbf{A}}} R_2$.

5 Generating Further Impossibilities

To extend our analysis, we now generate further impossibility results through Theorem 4.1. To do this, we consider various extension axioms employed in related literature. The first one is the dominance axiom used by Kelly (1977) in his analysis of strategy-proof social choice rules:

DOM. For any two distinct $X,Y\in\underline{\mathbf{A}}$ we have $X\ P\ Y$ whenever $x\ \rho\ y$ holds for all $x\in X$ and for all $y\in Y$.

We write δ for the consistency map determined by **DOM**. So, for every $\rho\in\Pi$, we have $\delta(\rho)=\{R\in\Re : R \text{ satisfies } \mathbf{DOM}\}$

A stronger axiom is the Gärdenfors (1976) principle defined as follows:

G. For any $X\in\underline{\mathbf{A}}$ and any $y\in\mathbf{A}\setminus X$ we have

(i) $X\ P\ X\cup\{y\}$ whenever $x^*\ \rho\ y$ where $x^*=\arg\min_X \rho$

(ii) $X\cup\{y\}\ P\ X$ whenever $y\ \rho\ x^*$ where $x^*=\arg\max_X \rho$

We write γ for the consistency map determined by **G**. So, for every $\rho\in\Pi$, we have $\gamma(\rho)=\{R\in\Re : R \text{ satisfies } \mathbf{G}\}$.

Finally, we have the modified version of the monotonicity axiom of Kannai and Peleg (1984), used by Roth and Sotomayor (1990):

M. For any $X\in 2^{\mathbf{A}}$, and $x,y\in\mathbf{A}\setminus X$ we have

$$X\cup\{x\}\ \ R\ \ X\cup\{y\} \text{ if and only if } x\ \rho\ y$$

We write μ for the consistency map determined by $\mathbf{M}$. So, for every $\rho \in \Pi$, we have $\mu(\rho) = \{R \in \Re : R \text{ satisfies } \mathbf{M}\}$.

We write $D^\delta = \bigcup_{\rho\in\Pi} \delta(\rho)$, $D^\mu = \bigcup_{\rho\in\Pi} \mu(\rho)$, $D^\gamma = \bigcup_{\rho\in\Pi} \gamma(\rho)$ for the domains determined by the respective consistency maps δ, μ and γ. It can be checked, as Kaymak and Sanver (2003) have shown, that the leximax and leximin extensions satisfy all three conditions. Hence, D^λ is a subset of D^δ, D^μ and D^γ.

Theorem 5.1. Consider a unanimous hyperfunction $f : [D]^{\mathbf{N}} \to \underline{\mathbf{A}}$

(i) Let $D = D^\delta$. f is strategy-proof if and only if f is dictatorial.

(ii) Let $D = D^\gamma$. f is strategy-proof if and only if f is dictatorial.

(iii) Let $D = D^\delta \cap D^\mu$. f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

(iv) Let $D = D^\gamma \cap D^\mu$. f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof. We first show (i) and (ii). D^λ is a subset of D^δ and D^γ. Note that both D^δ and D^γ are fully regular. Moreover, each of them satisfies condition β^* but violates σ. Hence, by part (ii) of Theorem 4.1, a unanimous hyperfunction defined over these domains is strategy-proof if and only if it is dictatorial. To show (iii) and (iv), we start by noting that both $D^\delta \cap D^\mu$ and $D^\gamma \cap D^\mu$ are supersets of D^λ . Moreover they are fully regular while they satisfy conditions β^* and σ^*. So by part (i) of Theorem 4.1, a unanimous hyperfunction defined over these domains is strategy-proof if and only if it is dictatorial or bi-dictatorial. $\square$

We now consider domains restricted through expected utility consistency. A *utility function* $u : \mathbf{A} \to \mathbb{R}$ *represents* $\rho \in \Pi$ if $u(x) \geq u(y) \Leftrightarrow x \; \rho \; y$ for all $x, y \in \mathbf{A}$. A probability distribution over $X \in \underline{\mathbf{A}}$ is a mapping $\Omega^X : X \to [0,1]$ with the property that $\sum_{x\in X} \Omega^X(x) = 1$.

EUC. A preference ordering $R \in \Re$ is *expected utility consistent* (EUC) with $\rho \in \Pi$ if and only if given any $X, Y \in \underline{\mathbf{A}}$ with $X \; R \; Y$, there exists a utility function u representing ρ, and probability distributions Ω^X on X and Ω^Y on Y such that $\sum_{x\in X} \Omega^X(x)u(x) \geq \sum_{y\in Y} \Omega^Y(y)u(y)$.

For every $\rho \in \Pi$, let $\mathbf{EUC}(\rho) = \{R \in \Re : R \text{ is } EUC \text{ with } \rho\}$ and $D^{EUC} = \bigcup_{\rho\in\Pi} \mathbf{EUC}(\rho)$.

BEUC. A preference ordering $R \in \Re$ is *Bayesian expected utility consistent* (BEUC) with $\rho \in \Pi$ if and only if given any $X, Y \in \underline{\mathbf{A}}$ with $X\ R\ Y$, there exists a utility function u representing ρ and a probability distribution Ω on $\mathbf{A}$ such that

$$\sum_{x \in X} \frac{\Omega(x)}{\sum_{z \in X} \Omega(z)} u(x) \geq \sum_{y \in Y} \frac{\Omega(y)}{\sum_{z \in Y} \Omega(z)} u(y).$$

For every $\rho \in \Pi$, let $\mathbf{BEUC}(\rho) = \{R \in \Re : R \text{ is } BEUC \text{ with } \rho\}$ and $D^{BEUC} = \bigcup_{\rho \in \Pi} \mathbf{BEUC}(\rho)$.

UEUC. A preference ordering $R \in \Re$ is *uniform expected utility consistent* (UEUC) with $\rho \in \Pi$ if and only if given any $X, Y \in \underline{\mathbf{A}}$ with $X\ R\ Y$, there exists a utility function u representing ρ such that $\sum_{x \in X} \frac{1}{|X|} u(x) \geq \sum_{y \in Y} \frac{1}{|Y|} u(y)$.

For every $\rho \in \Pi$, let $\mathbf{UEUC}(\rho) = \{R \in \Re : R \text{ is } UEUC \text{ with } \rho\}$ and $D^{UEUC} = \bigcup_{\rho \in \Pi} \mathbf{UEUC}(\rho)$. Notice that $D^{UEUC} \subset D^{BEUC} \subset D^{EUC}$.

Theorem 5.2. Consider a unanimous hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$.

(i) Let $D = D^{EUC}$. f is strategy-proof if and only if f is dictatorial.

(ii) Let $D = D^{BEUC}$. f is strategy-proof if and only if f is dictatorial.

(iii) Let $D = D^{UEUC}$. f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof. First note that D^{λ} is a subset of D^{UEUC}, hence of D^{BEUC} and of D^{EUC}. Moreover, each of these domains is fully regular and satisfies condition β^*. To see (i) and (ii), it suffices to note that D^{EUC} and D^{BEUC} violate σ which, by part (ii) of Theorem 4.1, establishes that they are dictatorial. To see (iii), note that D^{UEUC} satisfies σ^* which, by part (i) of Theorem 4.1, proves that it is bi-dictatorial. $\square$

We close this section by relating our findings to those of BDS. For this purpose, we state the following two extension axioms, CEUC and CEUCEP, as they appear in BDS.

CEUC. A preference ordering $R \in \Re$ is *conditional expected utility consistent* (CEUC) with some $\rho \in \Pi$ if and only if there exists a utility function u representing ρ and a probability distribution Ω on $\mathbf{A}$ such that

$$X\ R\ Y \Leftrightarrow \sum_{x \in X} \frac{\Omega(x)}{\sum_{z \in X} \Omega(z)} u(x) \geq \sum_{y \in Y} \frac{\Omega(y)}{\sum_{z \in Y} \Omega(z)} u(y)$$

for all $X, Y \in \underline{\mathbf{A}}$.

For every $\rho \in \Pi$, let $\mathbf{CEUC}(\rho) = \{R \in \Re : R \text{ is } CEUC \text{ with } \rho\}$ and $D^{CEUC} = \bigcup_{\rho\in\Pi} \mathbf{CEUC}(\rho)$.

CEUCEP. A preference ordering $R \in \Re$ is *conditional expected utility consistency with equal probabilities* (CEUCEP) with some $\rho \in \Pi$ if and only if there exists a utility function u representing ρ such that

$$X \; R \; Y \Leftrightarrow \sum_{x\in X} \frac{1}{|X|} u(x) \geq \sum_{y\in Y} \frac{1}{|Y|} u(y)$$

for all $X, Y \in \underline{\mathbf{A}}$.

Similarly, for every $\rho \in \Pi$, let $\mathbf{CEUCEP}(\rho) = \{R \in \Re : R \text{ is } CEUCEP \text{ with } \rho\}$ and $D^{CEUCEP} = \bigcup_{\rho\in\Pi} \mathbf{CEUCEP}(\rho)$.

It immediately follows from the definitions that $D^{CEUC} \subset D^{BEUC}$ and $D^{CEUCEP} \subset D^{UEUC}$. The set inclusions are proper and as Can et al. (2008) have shown, D^{λ} is neither a subset of D^{CEUC} nor of D^{CEUCEP}. So, Theorem 4.1 does not immediately yield the BDS results. However, with further elaboration, it is possible to generate BDS impossibilities through Theorem 4.1 for social choice correspondences. Formally speaking, a *social choice correspondence* (SCC) is a mapping $f : \Pi^{\mathbf{N}} \to \underline{\mathbf{A}}$. Note that the set of SCCs coincides with the set of hyperfunctions which satisfy the following invariance property:

A hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ is *invariant* if for any $\underline{R}, \underline{R}' \in D^{\mathbf{N}}$ with $\{x\} \; R_i \; \{y\} \Leftrightarrow \{x\} \; R_i' \; \{y\}$ for all $i \in \mathbf{N}$ and $x, y \in \mathbf{A}$, we have $f(\underline{R}) = f(\underline{R}')$.

Proposition 5.1. An invariant hyperfunction $f : [D^{BEUC}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ is strategy-proof if and only if f is strategy-proof over D^{CEUC}.

Proof. The "only if" part directly follows from the definitions. To see the "if" part, take an invariant hyperfunction $f : [D^{BEUC}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ which is not strategy-proof. So, there exists $i^* \in \mathbf{N}$ and $\underline{R}, \underline{R}' \in [D^{BEUC}]^{\mathbf{N}}$ with $R_j = R_j'$ for all $j \in \mathbf{N} \diagdown \{i^*\}$ such that $f(\underline{R}') \; P_{i^*} \; f(\underline{R})$. Now, pick $\underline{Q}, \underline{Q}' \in [D^{CEUC}]^{\mathbf{N}}$ such that $Q_j = Q_j'$ for all $j \in \mathbf{N} \diagdown \{i^*\}$ while $\{x\} \; Q_i \; \{y\} \Leftrightarrow \; \{x\} \; R_i \; \{y\}$ and $\{x\} \; Q_i' \; \{y\} \Leftrightarrow \; \{x\} \; R_i' \; \{y\}$ hold for all $i \in \mathbf{N}$, for all $x, y \in \mathbf{A}$, and the set $f(\underline{R}')$ is preferred to the set $f(\underline{R})$ under the relation Q_{i^*}. Existence of these preference relations is ensured by the definition of CEUC and BEUC. Therefore, as f is invariant, we have $f(\underline{Q}) = f(\underline{R})$ and $f(\underline{Q}') = f(\underline{R}')$. Thus, i^* can manipulate f at Q, implying that f is not strategy-proof over D^{CEUC}. $\square$

The arguments in the proof of Proposition 5.1 show, mutatis mutandis, the following proposition which we state with no formal proof:

Proposition 5.2. An invariant hyperfunction $f : [D^{UEUC}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ is strategy-proof if and only if f is strategy-proof over D^{CEUCEP}.

It is important to remark that Propositions 5.1 and 5.2 generate BDS results for social choice correspondences.

Theorem 5.3. Consider a unanimous and invariant hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$.

(i) Let $D = D^{CEUC}$. f is strategy-proof if and only if f is dictatorial.

(ii) Let $D = D^{CEUCEP}$. f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof. Take a unanimous and invariant hyperfunction $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$. We first show (i). The "if" part is obvious. To see the "only if" part, let $f : [D^{CEUC}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ be strategy-proof. Take any invariant hyperfunction $\varphi : [D^{BEUC}]^{\mathbf{N}} \to \underline{\mathbf{A}}$ such that $\varphi(\underline{R}) = f(\underline{R})$ for all $\underline{R} \in [D^{CEUC}]^{\mathbf{N}}$. Then, φ is strategy-proof by Proposition 5.1 and dictatorial by Theorem 5.2. Hence, f is dictatorial. Similar arguments prove (ii). $\square$

6 Concluding Remarks

We establish a general impossibility result on strategy-proof hyperfunctions. We identify a domain D^λ of lexicographic orderings over which a result of the Gibbard-Satterthwaite type prevails. Moreover, almost all superdomains of D^λ inherit this impossibility. It may also be worth recalling that any impossibility of the Gibbard-Satterthwaite type obtained for hyperfunctions can immediately be translated into the standard world of social choice correspondences. Thus, our main result expressed in Theorem 4.1 implies or paves the way to explain many previous results - hence the qualification "general" appearing in the title.

We wish to conclude by giving a more detailed discussion of the literature related to our paper. We start by comparing our results with those on strategy-proof hyperfunctions. BDS is the paper which is certainly closer to our analysis. It makes sense to say that D^λ extracts the essential structure of the BDS domains leading to the impossibility of the Gibbard-Satterthwaite type. In fact, the proof of our Theorem 3.1 is based on the proof of their main result. On the other hand, our and their domains are not subsets of each other. So, our results are logically independent. Nevertheless, with additional arguments, Theorem 4.1 generates the BDS impossibility for invariant hyperfunctions.

Another analysis of strategy-proof hyperfunctions is made by Benoit (2002). Our frameworks are quite similar. By using a near-unanimity condition which implies unanimity, he establishes an impossibility over domains containing orderings that are compatible with leximin extensions. Our results and those of Benoit (2002) are for different domains, hence being logically independent.

We also wish to compare our results with those on strategy-proof social choice correspondences. The typical analysis is made by considering a social choice correspondence $F : \Pi^{\mathbf{N}} \rightarrow \underline{\mathbf{A}}$ under a consistency map κ. We say that F is strategy-proof under κ if and only if given any $\underline{\rho} \in \Pi^{\mathbf{N}}$, $i \in \mathbf{N}$ and $\underline{\rho}' \in \Pi^{\mathbf{N}}$ with $\rho_j = \rho'_j$ for all $j \in \mathbf{N} \diagdown \{i\}$, we have $F(\underline{\rho})\ R_i\ F(\underline{\rho}')$ for all $R_i \in \kappa(\rho_i)$. Note that the non-existence of strategy-proof hyperfunctions defined over the domain $\Re^\kappa = \bigcup_{\rho \in \Pi} \kappa(\rho)$ implies the non-existence of strategy-proof social choice correspondences under κ. Thus, many previous impossibility results on strategy-proof social choice correspondences can be obtained through our theorems. We use this fact as an opportunity to remark that although the proof of Theorem 3.1 is based on the BDS proofs, our ability of generalizing this result to superdomains of D^λ provides a new insight. Theorem 4.1 covers a variety of domains consistent with expected utilities as well as the environments defined by the extension axioms which Gardenfors (1976), Barberà (1977) and Kelly (1977) use in their analysis of strategy-proof social choice correspondences.[9]

In particular, a more recent analysis made by Ching and Zhou (2002) also falls in this category. They consider manipulability of social choice correspondences assuming "Bayesian rational" individuals. Their Lemma 1 shows it to be equivalent to a consistency map determined by the Gärdenfors (1976) principle. In this framework, they characterize strategy-proof social choice correspondences as dictatorial or constant ones - an impossibility result which almost follows from part (ii) of Theorem 5.1.[10] Another related work is by Duggan and Schwartz (2000) who analyze the manipulability of social choice correspondences under a consistency map determined by expected utility consistency. Their result essentially follows from part (i) of Theorem 5.2.[11]

To sum up, we know through Theorem 4.1 that almost all regular superdomains of D^λ which satisfy a very weak condition β^* are dictatorial or bi-dictatorial depending on the satisfaction or failure of a certain separability condition.[12] Moreover, most of the standard extension axioms of the literature are compatible with the leximax and leximin extensions, leading to domains which contain D^λ. These axioms are compatible with the "non-resolute outcome" interpretation of a set. However, as Ozyurt and Sanver (2008) show, similar impossibilities arise under axioms compatible with the "resolute outcome" interpretation of a set as well. Thus, it is possible to interpret the current picture as the essential impossibility of escaping the Gibbard-Satterthwaite theorem for set-valued social choice rules.

[9]Some of these papers establish existence results under a weak definition of strategy-proofness where a social choice correspondence F is strategy-proof under κ if and only if given any $\underline{\rho} \in \Pi^{\mathbf{N}}$, $i \in \mathbf{N}$ and $\underline{\rho}' \in \Pi^{\mathbf{N}}$ with $\rho_j = \rho'_j$ for all $j \in \mathbf{N} \setminus \{i\}$, we have $F(\underline{\rho})\ R_i\ F(\underline{\rho}')$ for some $R_i \in \kappa(\rho_i)$.

[10]The qualification "almost" is needed because we impose the unanimity requirement over social choice correspondences which rules out constant ones.

[11]The qualification "essentially" is needed because Duggan and Schwartz (2000) consider social choice correspondences that map preference profiles over alternatives into a countable non-empty set while we assume finiteness.

[12]We say "almost", as Theorem 4.1 derives no conclusion for the domains which violate σ^* but satisfy σ. Nevertheless, we have not encountered any such domain in the literature.

APPENDIX A

Theorem 3.1. A unanimous hyperfunction $f : [D^\lambda]^{\mathbf{N}} \to \underline{\mathbf{A}}$ is strategy-proof if and only if f is dictatorial or bi-dictatorial.

Proof. We prove Theorem 3.1 by benefiting from BDS who prove a similar result for a domain D_E of admissible orderings defined through the idea of conditional expected utility consistency with equal probabilities. The claim that Theorem 3.1 makes on D^λ is exactly the same as their Theorem 3.3 makes on D_E.

It is critical to remark that Lemma 3.5 of BDS identifies four properties of D_E. These are as follows:[13]

(i) For all $P \in D_E$ and for all $x, y \in \mathbf{A}$, we have $\{x\}\ P\ \{y\} \ \Rightarrow\ \{x\}\ P\ \{x,y\}\ P\ \{y\}$.

(ii) For all $P \in D_E$, $X \in \underline{\mathbf{A}}$ and $x, y \in \mathbf{A}\diagdown X$ we have $X \cup \{x\}\ P\ X \cup \{y\} \Leftrightarrow \{x\}\ P\ \{y\}$.

(iii) For all $x, y \in \mathbf{A}$, there exists $P \in D_E$ such that $\arg\max_{\underline{\mathbf{A}}} P = \{x\}$ and $\{x\}\ P\ \{x,y\}\ P\ \{y\}\ P\ X$ for all $X \in \underline{\mathbf{A}}\diagdown\{\{x\},\{x,y\},\{y\}\}$.

(iv) For any set $X = \{b_1, b_2, \cdots, b_L\} \in \underline{\mathbf{A}}$ with $|L| \geq 3$ and $Y \in \underline{\mathbf{A}}$ which is distinct from X, there exists $P \in D_E$ with $\arg\max_{\underline{\mathbf{A}}} P = \{b_1\}$ while $\{x\}\ P\ \{b_L\}$ for all $x \in X$ such that $X\ P\ \{b_1, b_L\}\ P\ Y$ whenever $|Y| \geq L$ or $Y = \{a, b_L\}$ where $a \in \mathbf{A}\diagdown\{b_1\}$.

These are the properties of D_E that allow BDS to prove their Theorem 3.3 for the case $|\mathbf{N}| = 2$. The induction step in which they generalize their result to the case $|\mathbf{N}| > 2$ uses the existence of the following two additional orderings:[14]

[13] Properties (i), (ii), (iii) and (iv) below correspond respectively to the conditions (i), (ii), (iv) and (v) of their Lemma 3.5. Note that condition (iii) of Lemma 3.5 is not about D_E.

[14] One can see BDS to remark that (v) is used to handle "Case 2" in pages 388-389 while (vi) corresponds to the two orderings used in page 390. Note that (vi), as we state here, is different than the orderings required in page 390 of BDS. For, the BDS requirements, as they appear, are not expected utility consistent. What we state here as (vi) is the "corrected version", as expressed in the corrigendum of BDS. We also wish to draw attention to the last two sentences of Section 4 in BDS (p. 393) where they say: "The only assumptions on preferences that we require are specified completely in Lemma 3.5" We emphasize that the statement is true for proving their Theorem 3.3 for the 2-person case. On the other hand, the general proof requires the six orderings we specify.

(v) For all $x, y \in \mathbf{A}$, there exists $P \in D_E$ such that $X\ P\ \{y\}\ P\ \{x,y\}\ P\ \{x\}$ for all $X \in \underline{\mathbf{A}} \setminus \{\{x\}, \{x,y\}, \{y\}\}$.

(vi) For all $x, y, z \in \mathbf{A}$, there exists $P \in D_E$ such that $\{x\}\ P\ \{x,y\}\ P\ \{y\}\ P\ \{x,z\}\ P\ \{x,y,z\}\ P\ \{y,z\}\ P\ \{z\}\ P\ X$ for all $X \in \mathbf{A} \setminus \{\{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}\ \{y,z\}, \{x,y,z\}\}$

As a result, we prove our Theorem 3.1 by checking that D^λ also satisfies these six properties. In other words, we show that D_E can be replaced by D^λ in these six conditions. In fact, both the leximax and leximin extensions satisfy the conditions imposed by (i) and (ii). Thus, D_E can be replaced by D^λ in (i) and (ii). The existence of the ordering required by (iii) is ensured by the leximin extension. In other words, in (iii), D_E can be replaced by D^{λ^-} hence D^λ. The leximax extension ensures the existence of the orderings required by (iv) and (v). So in both (iv) and (v), D_E can be replaced by D^{λ^+} hence D^λ. Finally, the leximin extension ensures the existence of the ordering required by (vi). So in (vi), D_E can be replaced by D^{λ^-} hence D^λ, which completes our proof. $\square$

APPENDIX B

Proposition 4.2. Let $f : D^{\mathbf{N}} \to \underline{\mathbf{A}}$ be a strategy-proof hyperfunction with $D \supset D^{\lambda}$ being a (fully regular) domain satisfying conditions β^* and σ^*. If $i, j \in \mathbf{N}$ are the bi-dictators of f over D^{λ} then i and j are the bi-dictators of f over D.

Proof. Take D and f as in the statement of the proposition. Assume, without loss of generality that the bi-dictators over D^{λ} are agents 1 and 2. We will prove the proposition through two lemmata. Given any $\underline{R} \in [D]^{\mathbf{N}}$, $i \in N$ and $R_i \in D$, we write $(R_i, \underline{R}^{-i}) \in [D]^{\mathbf{N}}$ for the profile where there preference of all agents but i are as in $\underline{R}$ while R_i is the preference of i.

Lemma B1. *For all $\underline{R} \in D^{\mathbf{N}}$ with $R_1, R_2 \in D^{\lambda}$ we have $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1 \cup \arg\max_{\underline{\mathbf{A}}} R_2$.*

Proof. Take any $\underline{R} \in D^{\mathbf{N}}$ with $R_1, R_2 \in D^{\lambda}$. Write $k = |\{i \in \mathbf{N} \diagdown \{1, 2\} : R_i \in D \diagdown D^{\lambda}\}$. We claim that $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1 \cup \arg\max_{\underline{\mathbf{A}}} R_2$ for any value of $k \in \{1, \cdots, n-2\}$. We prove our claim by induction on k. We present the proof only for $k = 1$ while these arguments, mutatis mutandis, establish the inductive step. So take $k = 1$ and let, without loss of generality, $R_3 \in D \diagdown D^{\lambda}$. We want to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \arg\max_{\underline{\mathbf{A}}} R_1$ and $\{a_2\} = \arg\max_{\underline{\mathbf{A}}} R_2$. Suppose for a contradiction that $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \diagdown \{\{a_1, a_2\}\}$.

If $X \neq \{a_2\}$, then agent 3 manipulates at $(R_3', \underline{R}^{-3}) \in D^{\mathbf{N}}$ where $R_3' \in D^{\lambda}$ with $Y\ P_3'\ \{a_1\}\ P_3'\ \{a_1, a_2\}\ P_3'\ \{a_2\}$ for all $Y \in \underline{\mathbf{A}} \diagdown \{\{a_1\}, \{a_1, a_2\}, \{a_2\}\}$ by pretending $R_3 \in D \diagdown D^{\lambda}$, since $X\ P_3'\ \{a_1, a_2\}$. This argument, mutatis mutandis, shows the manipulability of f when $X \neq \{a_1\}$. As $X \neq \{a_2\}$ and $X \neq \{a_1\}$ exhaust all possible cases, this completes the proof of Lemma B1. □

Lemma B2. *For all $\underline{R} \in D^{\mathbf{N}}$ with $R_1 \in D^{\lambda} \Leftrightarrow R_2 \in D \diagdown D^{\lambda}$ we have $f(\underline{R}) = \arg\max_{\underline{\mathbf{A}}} R_1 \cup \arg\max_{\underline{\mathbf{A}}} R_2$.*

Proof. Take any $\underline{R} \in D^{\mathbf{N}}$ as in the statement of the lemma. Assume without loss of generality that $R_1 \in D \diagdown D^{\lambda}$ while $R_2 \in D^{\lambda}$. We want to show that $f(\underline{R}) = \{a_1, a_2\}$ where $\{a_1\} = \arg\max_{\underline{\mathbf{A}}} R_1$ and $\{a_2\} = \arg\max_{\underline{\mathbf{A}}} R_2$. Suppose not, i.e. $f(\underline{R}) = X$ for some $X \in \underline{\mathbf{A}} \diagdown \{\{a_1, a_2\}\}$. We consider three cases that are exhaustive about the value of X:

Case 1. $|X| = 1$

If $X = \{x\}$ for some $x \in \mathbf{A} \diagdown \{a_2\}$ then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{x\}$ by pretending R_1, since $\{x\}\ P'_1\ \{x, a_2\}$. If $X = \{a_2\}$, then agent 1 manipulates f at $\underline{R}$ by pretending some $R'_1 \in D^{\lambda}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$, since $\{a_1, a_2\}\ P_1\ \{a_2\}$.

Case 2. $|X| = 2$

If $X = \{a_1, x\}$ for some $x \in \mathbf{A} \diagdown \{a_1, a_2\}$ then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{x\}$ and $\{x\}\ P'_1\ \{a_1\}\ P'_1\ \{a_2\}$ by pretending R_1, since $X\ P'_1\ \{x, a_2\}$. If $X = \{a_2, x\}$ for some $x \in \mathbf{A} \diagdown \{a_1, a_2\}$ then agent 1 manipulates f at $\underline{R}$ by pretending some $R'_1 \in D^{\lambda}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$, as $\{a_1, a_2\}\ P_1\ X$ by D satisfying condition σ^*. If $X = \{x, y\}$ for some $x, y \in \mathbf{A} \diagdown \{a_1, a_2\}$ then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda^-}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$ and $\{a_1\}\ P'_1\ \{x\}\ P'_1\ \{y\}\ P'_1\ \{a_2\}$ by pretending R_1, since $\{x, y\}\ P'_1\ \{a_1, a_2\}$.

Case 3. $|X| > 2$

If $a_1 \in X$ but $a_2 \notin X$ then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$ and $\arg\min_{\underline{\mathbf{A}}} R'_1 = \{a_2\}$ by pretending R_1, since $X\ P'_1\ \{a_1, a_2\}$. If $a_2 \in X$ but $a_1 \notin X$, then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda^+}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{x\}$ for some $x \in X \diagdown \{a_2\}$ while $\arg\min_{\underline{\mathbf{A}}} R'_1 = \{a_2\}$ by pretending R_1, since $X\ P'_1\ \{x, a_2\}$. If $a_1, a_2 \in X$ then agent 1 manipulates f at$(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda^+}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$ and $\arg\min_{\underline{\mathbf{A}}} R'_1 = \{a_2\}$ by pretending R_1, since $X\ P'_1\ \{a_1, a_2\}$. If $a_1, a_2 \notin X$ then agent 1 manipulates f at $(R'_1, \underline{R}^{-1}) \in D^{\mathbf{N}}$ where $R'_1 \in D^{\lambda^-}$ with $\arg\max_{\underline{\mathbf{A}}} R'_1 = \{a_1\}$ and $\arg\min_{\underline{\mathbf{A}}} R'_1 = \{a_2\}$ by pretending R_1, since $X\ P'_1\ \{a_1, a_2\}$, completing the proof of Lemma B2.

□

The arguments used in the proof of Lemma B2 along with Lemma B1 and B2 suffice to prove the claim of Proposition 4.2 □

APPENDIX C

Theorem 4.1. Take any fully regular domain $D \supseteq D^\lambda$ satisfying condition β^* and consider a unanimous hyperfunction $f : [D]^{\mathbf{N}} \to \underline{\mathbf{A}}$.

(i) If D satisfies condition σ^*, then f is strategy-proof if and only if f is dictatorial or bi-dictatorial.

(ii) If D violates condition σ, then f is strategy-proof if and only if f is dictatorial.

Proof. Take any fully regular domain $D \supseteq D^\lambda$ satisfying condition β^* and a unanimous hyperfunction $f : [D]^{\mathbf{N}} \to \underline{\mathbf{A}}$. We first show (i). Let D satisfy condition σ^*. To see the "if" part, let f be dictatorial or bi-dictatorial. If f is dictatorial then f is trivially strategy-proof. If f is bi-dictatorial then, by Proposition 4.3, it is strategy-proof. To see the "only if" part, assume f is strategy-proof over D. So f is strategy-proof over D^λ as well. Hence by Theorem 3.1, f is dictatorial or bi-dictatorial over D^λ. If f is dictatorial over D^λ then, by Proposition 4.1, it will be dictatorial over D as well. If f is bi-dictatorial over D^λ then, by Proposition 4.2, it will be bi-dictatorial over D.

To show (ii), let D violate σ. The "if" part is clear, as f being dictatorial implies its strategy-proofness. To see the "only if" part, assume f is strategy-proof over D, hence over D^λ as well. Therefore, by Theorem 3.1, f is either dictatorial or bi-dictatorial over D^λ. If f is dictatorial over D^λ then, by Proposition 4.1, f is dictatorial over D. We complete the proof by showing that f cannot be bi-dictatorial over D^λ while condition σ is violated. As D violate σ there exists $a, b, c \in \mathbf{A}$ and $R_1^* \in D \diagdown D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1^* = \{a\}$ while $\{a\}\ P_1^*\ \{b\}\ P_1^*\ \{c\}$ such that $\{b, c\}\ P_1^*\ \{a, c\}$. Now, suppose for a contradiction that f is bi-dictatorial over D^λ. Let, without loss of generality, agents 1 and 2 be the bi-dictators. Pick some $R_2^* \in D^{\lambda^-}$ with $\arg\max_{\underline{\mathbf{A}}} R_2^* = \{c\}$ and $\{c\}\ P_2^*\ \{a\}\ P_2^*\ \{x\}$ for all $x \in \mathbf{A} \diagdown \{a, c\}$. We write (R_1^*, R_2^*) for the preference profile where agents 1 and 2 have the respective orderings R_1^* and R_2^* while each of the remaining agents has some ordering in D^λ. From now on, we fix these orderings of the remaining agents which are inessential for the proof (as long as they belong to D^λ) and define a preference profile by the orderings of the bi-dictators.

Let $f(R_1^*, R_2^*) = X$. The following three cases, each of which gives a contradiction, exhaust the value that X can take, thus complete the proof.

Case 1. $|X| = 1$

If $X = \{x\}$ for some $x \in \mathbf{A} \setminus \{c\}$, then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{x\}$ by pretending R_1^*, as $\{x\}\ P_1\ \{x, c\}$. If $X = \{c\}$, then agent 1 manipulates f at (R_1^*, R_2^*), by pretending some $R_1 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{a\}$ since $\{a, c\}\ P_1^*\ \{c\}$ as D satisfies condition β^*.

Case 2. $|X| = 2$

If $X = \{a, x\}$ for some $x \in \mathbf{A} \setminus \{a, c\}$, then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{x\}$ and $\{x\}\ P_1\ \{a\}\ P_1\ \{c\}$ by pretending R_1^*, as $\{a, x\}\ P_1\ \{x, c\}$. If $X = \{c, x\}$ for some $x \in \mathbf{A} \setminus \{a, c\}$, then agent 2 manipulates f at (R_1^*, R_2^*) by pretending some $R_2 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_2 = \{a\}$, as $f(R_1^*, R_2) = \{a\}$ by the unanimity of f while $\{a\}\ P_2^*\ \{c, x\}$. If $X = \{x, y\}$ for some $x, y \in \mathbf{A} \setminus \{a, c\}$ then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^{\lambda^-}$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{a\}$ while $\{a\}\ P_1\ \{x\}\ P_1\ \{y\}\ P_1\ \{c\}$ by pretending R_1^*, as $\{x, y\}\ P_1\ \{a, c\}$. If $X = \{a, c\}$ then agent 1 manipulates f at (R_1^*, R_2^*) by pretending some $R_1 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{b\}$, as $\{b, c\}\ P_1^*\ \{a, c\}$

Case 3. $|X| > 2$

If $a \in X$ but $c \notin X$ then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^\lambda$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{a\}$ and $\arg\min_{\underline{\mathbf{A}}} R_1 = \{c\}$ by pretending R_1^*. If $c \in X$ but $a \notin X$ then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^{\lambda^+}$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{x\}$ for some $x \in X \setminus \{c\}$ while $\arg\min_{\underline{\mathbf{A}}} R_1 = \{c\}$ by pretending R_1^*. If $a, c \notin X$, then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^{\lambda^-}$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{a\}$ while $\arg\min_{\underline{\mathbf{A}}} R_1 = \{c\}$ by pretending R_1^*. If $a, c \in X$, then agent 1 manipulates f at any (R_1, R_2^*) where $R_1 \in D^{\lambda^+}$ with $\arg\max_{\underline{\mathbf{A}}} R_1 = \{a\}$ while $\arg\min_{\underline{\mathbf{A}}} R_1 = \{c\}$ by pretending R_1^*.

$\square$

REFERENCES

Barberà, S. (1977), "The Manipulability of Social Choice Mechanisms that do not Leave Too Much to Chance", Econometrica, 45: 1572-1588

Barberà, S., B. Dutta, and A. Sen (2001), "Strategy-Proof Social Choice Correspondences", Journal of Economic Theory, 101: 374-394.

Barberà, S. and P. K. Pattanaik (1984), "Extending an Order on a Set to the Power Set: Some Remarks on Kannai-Peleg's Approach", Journal of Economic Theory, 32: 185-191.

Benoit, J. P. (2000), "The Gibbard-Sattertwaite Theorem: A Simple Proof", Economic Letters, 69: 319-322.

Benoit, J. P. (2002), "Strategic Manipulation in Voting Games When Lotteries and Ties are Permitted", Journal of Economic Theory, 102, 421-436.

Can, B., B. Erdamar, and M.R. Sanver (2008), "Expected Utility Consistent Extensions of Preferences", Theory and Decision, Forthcoming.

Campbell, D. and J. S. Kelly (2002), "A Leximin Characterization of Strategy-Proof and Non-Resolute Social Choice Procedures", Economic Theory, 20: 809-829.

Ching, S. and L. Zhou (2002), "Multi-Valued Strategy-Proof Social Choice Rules", Social Choice and Welfare, 19: 569-580.

Duggan, J. and T. Schwartz (2000), "Strategic Manipulability without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized", Social Choice and Welfare, 17: 85-93.

Feldman, A. (1979a): "Nonmanipulable Multi-Valued Social Decision Functions", Public Choice, 34: 177-188

Feldman, A. (1979b), "Manipulation and the Pareto Rule", Journal of Economic Theory, 21: 473-482.

Feldman, A. M. (1980), "Strongly Nonmanipulable Multi-Valued Collective Choice Rules", International Review for Social Sciences, 33

Gärdenfors, P. (1976), "Manipulation of Social Choice Functions", Journal of Economic Theory, 13: 217-228.

Gärdenfors, P. (1978), "On Definitions of Manipulation of Social Choice Functions", in J.J. Laffont (ed.), Aggregation and Revelation of preferences,. Amsterdam: North Holland.

Gibbard, A. (1973), "Manipulation of Voting Schemes: A General Result", Econometrica, 41, 587-601.

Kannai Y. and B. Peleg (1984), "A Note on the Extension of an Order on a Set to the Power Set", Journal of Economic Theory, 32: 172-175

Kaymak, B. and M. R. Sanver (2003), "Sets of Alternatives as Condorcet Winners", Social Choice and Welfare, 20 (3), 477-494.

Kelly, J. (1977), "Strategy-Proofness and Social Choice Functions without Single-Valuedness", Econometrica, 45: 439-4

Özyurt, S. and M.R. Sanver (2008), "Strategy-Proof Resolute Social Choice Correspondences", Social Choice and Welfare, 30(1), 89-101.

Pattanaik, P.K. (1973), "On the Stability of Sincere Voting Situations", Journal of Economic Theory, 6: 558-574.

Roth, A. and M. A. O. Sotomayor (1990), "Two-Sided Matching: A Study in Game Theoretic Modelling and Analysis", Cambridge University Press.

Sanver, M. R. (2007), "A Characterization of Superdictatorial Domains for Strategy -Proof Social Choice Functions", Mathematical Social Sciences, 54(3), 257-260.

Satterhwaite, M. (1975), "Strategy-proofness and Arrow's Conditions : Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", Journal of Economic Theory. 10, 187-217.