Strategy-proof and Efficient Mediation: An Ordinal Market Design Approach*

Onur Kesten† Selçuk Özyurt‡

April 2020

Abstract

Mediation, also known as assisted negotiation, is the preferred alternative dispute resolution approach that has given rise to a multi-billion-dollar industry worldwide. Online dispute resolution providers often rely on mechanized mediation procedures. We develop a novel ordinal market design approach where negotiators with conflicting preferences seek resolution over (at least) two issues. The mediation process is represented by mechanisms with voluntary participation. We characterize the full class of efficient, individually rational, and strategy-proof mediation protocols. A necessary and sufficient condition for the existence of such protocols is the so-called quid pro quo property that allows negotiators to compromise between issues.

JEL Codes: C78; D47; D74; D78; D82

Keywords: Market design, alternative dispute resolution, mediation, online dispute resolution, negotiation, logrolling, quid pro quo, outside options, strategy-proofness, efficiency

*We thank Hülya Eraslan, Johannes Hörner, Mehmet Ekmekçi, Tayfun Sönmez, Rakesh Vohra, George Mailath, Vijay Krishna, Hervé Moulin, Utku Ünver, Larry Ausubel, Michael Ostrovsky, Ed Green, Ron Siegel, Luca Rigotti, Sevgi Yüksel, Alexey Kushnir, Alex Teytelboym, William Thomson, Peter Troyan, Charlie Holt, Ruben Juarez, Francis Bloch, Leeat Yariv, Laura Doval, Piotr Dworczak, Remzi Sanver, Nicholas Yannelis, and all seminar participants at Stanford (NBER Market Design Workshop), NYU, Rice, Maryland, Boston College, University of Pittsburgh, Carnegie Mellon, Penn State (PETCO), Virginia, Durham, Sydney, University of Technology Sydney, Bilkent, ITU, Paris Dauphine, York, Dalhoise, Sussex, and SAET for many useful discussions and suggestions. Selçuk Özyurt specially thanks Harvard, Carnegie Mellon and Sabancı Universities for their support and hospitality during this project, and European Commission for the funding from European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 659780.

†University of Sydney and Carnegie Mellon University, Tepper Business School 5000 Forbes Ave. Pittsburgh, PA 15213 (okesten@andrew.cmu.edu)

‡Corresponding author. York University, Department of Economics, Room 1090, Vari Hall 4700 Keele Street Toronto, Ontario, M3J 1P3 (ozyurt@yorku.ca)

"*Now that society has embraced technology so thoroughly, the key question for dispute resolution professionals is, how can we leverage technology to best assist parties in resolving their disputes? Online Dispute Resolution (ODR) is no longer a novelty—it is now arguably the future of Alternative Dispute Resolution (ADR).*" Colin Rule[1]

1. Introduction

The bestseller *Getting to Yes*, by Roger Fisher and William Ury, is arguably one of the most famous—if not the most famous—works on the topic of negotiation. The authors identify conflict as a growth industry, and the last few decades have proved them right. Mediation is a consensual negotiation process in which a third party, known as the mediator, assists disputants/negotiators to identify the issues and assess the options and makes recommendations to reach an agreement short of litigation. Courts in all states in the U.S. and around the world currently offer some form of mediation for most cases.[2] Mediation is often the preferred form of alternative dispute resolution (ADR) approach due to its cost-effectiveness (time-wise and financially),[3] flexibility,[4] and confidentiality. A dispute that can take months or years in conventional court proceedings can be resolved in a few affordable mediation sessions. Employment, patent/copyright, business, interpersonal, and family conflicts are some of the most common types of mediated disputes. Dispute resolution centers of major online companies such as eBay, PayPal, Uber, and Amazon settle more than a billion disputes a year.

Notwithstanding the practical conveniences it affords, the mediation process is often considered less formal and less transparent than binding adjudication processes such as litigation and arbitration. Traditional legal theorists argue[5] that the low visibility and lack of formal rules and structure in mediation reduce the rights of less powerful participants.[6] A structured and rigorous view of mediation is pioneered in online dispute resolution (ODR) that often rely on automation and predetermined sets of rules. ODR systems are based on mechanized negotiation methods such as *double-blind-bidding* for single issues, which uses a type of an auction mechanism and *visual-blind-bidding* for multiple issues, which

[1]Co-Founder and Chief Operating Officer of Modria, an ODR provider in Silicon Valley. Colin Rule was the first Director of ODR at eBay/PayPal.

[2]Seventeen states in the U.S. require mandatory mediation: 6% of civil cases in Northern California courts in 2011, and 35.6% of civil and 21.6% of divorce cases in New York state courts in 2016 were mediated. The total value of mediated cases in the UK is estimated to be £10.5bn. in 2011, excluding mega-cases and family and community disputes.

[3]According to Hadfield (2000), it costs a minimum of $100,000 to litigate a straightforward business claim in the US, whereas a mediation session varies from few hours to a day and even the most reputable mediators charge around $10,000 - $15,000 for a day. In addition, disputants do not have to pay any fees for experts, witnesses, document preparation, investigation, or paralegal services, which would easily make the costs pile up.

[4]It is impossible to discuss a legally "irrelevant" issue in litigation/arbitration, and some disputes are not just about money or being right. In mediation, however, parties can discuss and negotiate issues that are not directly linked to the case. Empirical studies of mediation and program evaluations suggest a 60-90% success rate, a 90-95% satisfaction rate by the disputants, and a higher rate of compliance with mediator recommendations relative to court-imposed orders.

[5]See, for example, Damaska (1975).

[6]In a seminal work, LaFree and Rack (1996) provide empirical evidence from the small claims court mediation program in Bernalillo County in Albuquerque, New Mexico, and conclude that ethnicity and gender could be more important determinants in mediation than adjudication. In particular, they report that white males receive significantly more favorable outcomes in mediation than minority females. In a similar vein, many others emphasize the factors that can cause disputant dissatisfaction that are under the direct control of mediators. As a remedy, Tyler and Huo (2002) advocate the use of fair procedures that are described as those in which decisions are viewed as *neutral, objective, and consistent.*

allows parties to make proposal bundles, and an algorithm determines a mutually agreeable bundle among the proposals. At the final stage of ODR, the disputants accept or reject the proposed outcome. The ODR process can also be in the form of assisted negotiation as in the case of eBay's contractor SquareTrade, the leading ODR provider for consumer mediation. The SquareTrade dispute resolution process presents its claimant a list of possible solutions from which she selects the ones she agrees to. Upon agreeing to participate in the process, the other party is then asked to do the same. If a mutually agreeable resolution is selected, the dispute is resolved.[7]

In this paper, we take a market/mechanism design approach and search for neutral, efficient, and incentive compatible resolution recommendation mechanisms that suggest fruitful building blocks towards more systematic and consistent decision-making. Mechanism design has been successful in many applications, most notably in market design for auctions and matching. Unlike the traditional mechanism design approach to bargaining,[8] we adopt an *ordinal* approach in the context of mediation for three reasons. First, rather than restricting players' utilities to a specific transferable utility setting, we maintain a basic common implication of any monotonic preferences in a conflict situation: preferences of the negotiators over alternatives in a given issue are diametrically opposed.[9] In doing so, we characterize all classes of preferences that would support a possibility result and thereby allow for both transferable and nontransferable utility.[10] Second, it is genuinely simple to implement ordinal mechanisms, which is particularly important when agents are boundedly rational.[11] Third, the ordinal approach together with dominant strategy implementation makes it possible to avoid the famous Wilson critique by providing "detail-freeness" and "robust incentives" to participants.[12]

We assume that two negotiators are in a dispute/negotiation over two types of issues, a main issue and a second issue, and aim to reach a resolution through a mediator.[13] Negotiators come to the mediation table with a privately known "least acceptable outcome" in the main issue. Due to the informational asymmetry, the main issue exhibits *uncertain*

[7]Incidentally, this process has much in common with the central mechanism we propose and characterize in the paper.

[8]The traditional approach postulates that traders have private valuations drawn from pre-specified distributions and commonly known utility functions. This approach is pioneered by the seminal work of Myerson and Satterthwaite (1983). The Myerson-Satterthwaite setting corresponds to a single-issue mediation problem in our formulation and admits no strategy-proof and efficient mechanism. See the discussion in Section 6 where we demonstrate that adding an extra issue into the setting overturns this impossibility.

[9]We justify such modeling by showing that under a mild efficiency requirement, any situation where negotiators' preferences are not diametrically opposed can be equivalently represented as one where they are. See Section 2.

[10]Although money is an important issue in disputes, it is rarely the only issue (Malhotra and Bazerman, 2008). This underscores the necessity of a setting that can also admit nontransferable utility specifications.

[11]There is a large body of experimental evidence that finds that the representation of preferences by VNM utility functions may be inadequate; see, for example, Kagel and Roth (2016). This literature argues that the formulation of rational preferences over lotteries is a complex process that most agents prefer not to engage in if they can avoid it.

[12]While stressing the powerful insights that mechanism design offers in bargaining problems, Ausubel et al. (2002) voice a similar concern: "*... Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions.*"

[13]Although mediation is generally defined as a type of assisted negotiation via a neutral third party, we do not distinguish between assisted and non-assisted negotiations. For this reason, we use the terms "mediator" and "mechanism" interchangeably.

gains from mediation, i.e., whether a mutually acceptable resolution exists is unknown a priori. We capture such circumstances by assuming that each negotiator has an outside option whose ranking is her private information. Issues of this type are abundant in many examples of ADR such as family disputes (e.g., terms of child custody), dissolution of business partnership (e.g., buyout price), employment negotiation (e.g., job title or office location), etc. We say that an issue exhibits *certain gains from mediation* if it is common knowledge that a set of mutually acceptable resolutions exist. In wage negotiations, certain wage ranges can be assumed to be mutually acceptable for both the employee and the employer. Similarly, several examples of ADR such as family disputes (e.g., certain divisions of the assets such as a 50-50 split), tenancy disputes (e.g., mutually acceptable dates of vacating the property), and business disputes (e.g., non-monetary terms) cover issues of this type. If impasse is commonly known to be the least-preferred outcome for both negotiators, e.g., parties expect litigation to be a long and costly process, then we speak of certain gains from mediation since any mutually agreeable alternative Pareto dominates the outside option.

For an illustration, consider the following simple employment negotiation between an academic job candidate and a university in the UK mediated through the department head. There are two issues at the heart of the negotiation: job title and compensation. Possible job titles are Lecturer (L), Senior Lecturer (SL), Associate Professor (AP), Full Professor(P), i.e., $X = \{L, SL, AP, P\}$, and possible compensation levels are mandated by the university, publicly announced, and belong to the interval $Y = [£40K, £100K]$.[14] It is commonly understood that the university (candidate) prefers lower (higher) ranks and pay levels. We assume that the list of acceptable titles that are negotiable are each side's private information. Suppose, for example, that the department does not find the employee suitable for a Full Professor position whereas any title below Senior Lecturer is a "deal-breaker" for the candidate, i.e., job title is the main issue with uncertain gains from mediation.[15] On the other hand, both parties understand that once the title issue—main issue—is resolved, there are mutually agreeable compensation levels within the publicly announced pay range—second issue.[16]

Our modeling accommodates two possible interpretations of negotiators' outside options in the two issues. In the above example, a lack of agreement on the title makes the second issue vacuous. This type of a mediation problem corresponds to a case of *joint* outside options, e.g., if the two sides find no mutually agreeable alternative in the main issue, then there is automatically no resolution in the second issue, e.g., the outside option in the second issue is also implemented. An alternative interpretation corresponds to a case of *separate* outside options whereby the issues are independent and the mediator can propose a partial resolution in which the two sides agree on a mutually acceptable alternative in

[14] In the UK, the actual academic rankings are divided into several grades which are tied to the corresponding pay ranges. Our simplification in this illustration is without loss of generality.

[15] In this case, there is a potential gain from mediation on the title. By contrast, if, for example, any rank below Full Professor is unacceptable for the candidate, there is clearly no gain from mediation; hence the uncertainty in our characterization of this issue.

[16] Although monetary transfer (or, price) in this example is the second issue, there are many situations in which it is viewed as the main issue that has uncertain gains from mediation. Both scenarios are compatible with our model. We also allow for both issues to be non-monetary and discrete/continuous. See the Supplementary Appendix for examples.

one issue but not the other. This is analogous to addressing multiple issues in political bargaining; e.g., see Chen and Eraslan (2014, 2017). In the Supplementary Appendix, we provide an account of several actual mediation cases and discuss which interpretation of the model would be appropriate for each case.

The mediator's objective is to truthfully elicit negotiators' private information about how they rank their respective outside options and propose an efficient and mutually acceptable, i.e., individually rational, outcome. In the two-issue mediation problem,[17] the mediator recommends a bundle (x, y) of outcomes from two issues X and Y. A mediation rule is a systematic way of choosing an outcome for any reported pair of types of the two negotiators. The main question we ask is whether there is an impartial and dominant strategy incentive compatible, i.e., strategy-proof, way of soliciting true preferences so that mediation outcomes are always efficient and never vetoed.

A necessary and sufficient condition for obtaining a positive answer to our main question is the so-called *quid pro quo* property of negotiators' preferences. This assumption imposes a form of substitutability between issues X and Y. Quid pro quo entails preferences rich enough that a negotiator is able to make concessions in issue X for a more preferred alternative in issue Y, e.g., there are pairs of alternatives x and x' in X such that although x is preferred over x', there exists a corresponding pair of alternatives y and y' in Y and when bundled together, (x', y') is preferred over (x, y). Such preference reversals induce a partial order on the main issue, and we require that this partial order and (the connected subsets of) the main issue form a semilattice. In a classic exchange economy where alternatives represent quantities, for any given linear division plan, the condition is guaranteed by requiring preferences to be convex and continuous, e.g., quid pro quo is compatible with the CES and the quasi-linear utility.

Our main results are a complete characterization of the class of strategy-proof, efficient, and individually rational mediation rules and classes of preferences that admit such rules (Theorems 1 and 2). These rules operate through an exogenously specified precedence order over a special set of bundles, which we call the *logrolling bundles*, and choose the highest-precedence logrolling bundle among those that are mutually acceptable to both negotiators. As the precedence order varies, the characterized class of rules span what we refer to as the *family of logrolling rules*. A visual characterization of this family demonstrates that a rule f belongs to the family if and only if the lower half of the matrix representing f can be partitioned into rectangular regions, where each rectangle corresponds to the set of problems for which the same logrolling bundle is recommended (Theorem 3). The family of logrolling rules nests interesting special members. When the precedence order is in line with the preference ranking of a given negotiator over the logrolling bundles, we obtain the corresponding *negotiator-optimal rule*. A logrolling rule can also be implemented in a decentralized way as a "shortlisting rule" in the absence of an actual mediator: One of the negotiators offers a short list of bundles (including a disagreement bundle) to the other negotiator, and the other negotiator simply chooses her favorite bundle from this list.

A negotiator-optimal rule represents situations when a mediator may be categorically biased toward one side of the dispute. In keeping with our main objective of finding im-

[17] Our main model easily extends to the case of more than two issues. See Section 6.

partial mediation rules, we search for members of this class of rules that satisfy sensible fairness criteria. It turns out there is a central member of the family of strategy-proof, efficient, and individually rational mediation rules that satisfies an ordinal fairness criterion (Theorem 4). This is the so-called *constrained shortlisting* rule, which recommends the median logrolling bundle when it is mutually acceptable, and when it is not, the mutually acceptable logrolling bundle closest to it in precedence.

This paper introduces an ordinal and detail-free approach to mediation and negotiation.[18] As such, our paper spans literature on bargaining, two-sided assignment/matching, fair division, and political economy. The ordinal approach has been championed by its remarkable success in applications of matching and assignment such as medical residency, college admissions, school choice, and organ exchange, where a plethora of strategy-proof and efficient mechanisms have been proposed, extensively studied, and some even adopted in practice.[19] While such discrete resource allocation problems inspire the modeling in this paper, our approach readily extends to cardinal preferences and continuous/several issues as well.

Our setup shares important conceptual and mathematical parallels with two-sided matching and assignment models. Mathematically, the set of logrolling bundles coupled with a concatenation of the two sides' preferences forms a semilattice akin to the structure of the set of stable matchings in a two-sided market. Notably, a certain type of semilattice property proves both necessary and sufficient for attaining strategy-proofness in our model. As a consequence, one connection with matching models that surfaces is that we find the class of strategy-proof, efficient, and individually rational rules to contain the negotiator-optimal rules. These rules, though logically unrelated, bear resemblance to the proposing-side optimal deferred acceptance mechanisms in two-sided matching and the buyer/seller optimal core assignments in the Shapley-Shubik assignment game.[20] We provide a broader discussion of the related literature in Section 7.

The paper is organized as follows. Section 2 provides the preliminary insights with a simple example. Section 3 describes the main model and Section 4 provides the main results. In Section 5 we discuss special members of the logrolling family. Section 6 provides a discussion of the modeling assumptions and possible extensions. We discuss the related literature in Section 7 and conclude in Section 8. The proofs of our main results, i.e., Theorem 1 and 2, are provided in the Appendix. The Supplementary Appendix contains several examples of actual mediation cases that fit the model, the extension of the model to continuum of alternatives, the proof of the revelation principle, and all the other proofs.

[18]Our view should not be taken as one of universal endorsement of ordinal mechanisms over their cardinal counterparts, but rather as advocating their use in a specific context where the former can offer the practical convenience from a market design point of view. In a general mechanism design setting, Carroll (2018) shows that one loses generality by restricting to ordinal mechanisms. He also provides a foundation for ordinal mechanisms by showing that a planner can implement her goals robustly to uncertainty about cardinal preferences only if she uses an ordinal mechanism. Another limitation of ordinal mechanisms is that they cannot elicit preference intensities. See also Pycia (2014).

[19]See, for example, Gale and Shapley (1962), Shapley and Shubik (1971), Crés and Moulin (2001), and recent applications of ordinal assignment mechanisms such as Bogomolnaia and Moulin (2001), Abdülkadiroglu and Sönmez (2003), Budish (2011), and Ergin et al. (2017). In particular, our modeling of negotiator preferences is reminiscent to those in Crés and Moulin (2001), Bogomolnaia and Moulin (2004), and Roth, Sönmez and Ünver (2005).

[20]See Roth and Sotomayor (1990) for a survey of two-sided matching and assignment problems.

2. The Environment

To give the main insights we begin with a simple example showing the difficulty of constructing a strategy-proof mediation mechanism under uncertain gains from mediation. Then we discuss why a second issue with certain gains would help overcome this difficulty. This is followed by a brief discussion about why the assumption of diametrically opposed preferences is without loss of generality.

Single-issue mediation with uncertain gains: An impossibility

Negotiators 1 and 2 are in dispute over the division of a set of jointly owned assets. Let alternative x denote the percentage of the assets negotiator 1 gets, and so 2 gets the remaining $100 - x$ percent. Suppose, for simplicity, that $x_1 = 51$ and $x_2 = 49$ are the only possible alternatives for the parties. Negotiators are also entitled to an outside option, o, in case one or both of them reject the mediator's recommendation. The outside option could, for example, indicate the set of all possible outcomes of an alternative adjudication process. Therefore, the set $\overline{X} = \{x_1, x_2, o\}$ denotes the set of all possible outcomes of the dispute.

It is common knowledge that both negotiators prefer the larger share for herself, i.e., preferences over alternatives are diametrically opposed. Therefore, negotiator 1 prefers alternative x_1 to x_2 and negotiator 2 prefers x_2 to x_1. We assume that each negotiator ranks any "unacceptable" alternative below the outside option, which would be interpreted as the negotiator's "expected utility" of the outside option being higher than the utility of the unacceptable alternative. The ranking of the outside option, however, is the negotiators' private information, which would reflect, for example, the negotiators' private valuations or subjective beliefs about the possible outcomes of the adjudication process. Thus, each negotiator has two types:[21]

$\theta_1^{x_1}$	$\theta_1^{x_2}$
x_1	x_1
o	x_2
x_2	o

$\theta_2^{x_2}$	$\theta_2^{x_1}$
x_2	x_2
o	x_1
x_1	o

A mutually acceptable alternative does not always exist, and so, the dispute has uncertain gains from mediation. The mediation process is denoted by a mechanism f whose outcome can be vetoed by either negotiator, in which case both negotiators receive the outside option o. The mechanism f maps the negotiators' private information to an outcome in $\overline{X}$, and we use the following matrix to represent it:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$
$\theta_1^{x_1}$	$f_{1,1}$	$f_{1,2}$
$\theta_1^{x_2}$	$f_{2,1}$	$f_{2,2}$

where $f_{\ell,j} \in \overline{X}$ for all $\ell, j \in \{1,2\}$.

There is no mutually acceptable alternative for type profiles corresponding to the entry above the diagonal of the matrix, and thus, a self-enforcing mediation rule should suggest $f_{1,2} = o$. In such instances, the outside option is effectively the only result in all voluntary

[21] We assume, without loss of generality, that there is at least one acceptable alternative for each negotiator.

mediation processes (individual rationality). If the outcomes of the mediation process are (Pareto) efficient, then $f_{1,1}$ can be any outcome except the outside option. Moreover, if the process produces individually rational outcomes, then we must have $f_{1,1} = x_1$. Likewise, an efficient and individually rational mediation process must choose $f_{2,2} = x_2$ and $f_{2,1} \in \{x_1, x_2\}$. Namely, there are two mediation rules that satisfy individual rationality and efficiency.

However, neither of these rules is immune to strategic manipulation. To see this point, suppose that $f_{2,1} = x_1$. In this case, type $\theta_2^{x_1}$ of negotiator 2 would misreport her type as the less-accepting $\theta_2^{x_2}$ to obtain x_2, contradicting strategy-proofness. Symmetrically, if $f_{2,1} \neq x_1$, then type $\theta_1^{x_2}$ of negotiator 1 would declare her type as the less-accepting $\theta_1^{x_1}$ to obtain x_1, again contradicting strategy-proofness. Note that this impossibility prevails even when the mechanism is allowed to be stochastic.[22] It is straightforward to extend this impossibility to the case with more than two alternatives.

A practical method to relax the above tension among the desiderata is to introduce an additional issue. The idea is that the negotiators could then be asked to consider concessions in one issue for a favorable treatment in the other so they will be disincentivized from strategic preference truncations. In other words, the extra issue can serve the compensating role of money in a transferable utility model. However, it is not difficult to see that the impossibility carries over if each additional issue also exhibits uncertain gains from mediation. This is simply because it is impossible to guarantee that the new issue always contains mutually acceptable alternatives that can be used for compensation. This is summarized in the following remark, which is formalized in Section 6.

Remark 1: Consider a dispute with any number of issues each with at least two alternatives. Suppose that each negotiator has a private outside option in each issue such that she ranks any unacceptable alternative below that outside option in that issue. Then there is no strategy-proof, efficient, and individually rational mediation mechanism.

On the other hand, when the mediation problem has a second issue with a commonly known set of efficient alternatives, the mediator can leverage them to generate attractive "logrolling bundles" of alternatives that bring the two sides to the table with proper incentives. In particular, strategically being less-accepting (as in the above example) now entails forgoing a more preferred alternative from the second issue. Thus, provided that the second issue is sufficiently interesting for the negotiators, they would have incentive to report their types truthfully even if that means an inferior alternative in the first issue.

In standard mechanism design with transferable utility, it is well-known that there is no strategy-proof, efficient, individually rational, and budget balanced mechanism. Our possibility results reveal a parallel insight in the non-transferable utility world. To circumvent the impossibility, the mediator uses the second issue to either "inject funds" which happens in case of agreement, or "withhold funds" which happens in case of disagreement. In this sense, our mechanisms can be viewed analogously to Vickrey-Clarke-Groves mechanisms (Vickrey 1961, Groves 1973, and Clarke 1971) in an ordinal environment. In Section

[22] In that case, the only difference in the argument would be that $f_{2,1}$ would choose a lottery over x_1 and x_2. However, the above deviations would still remain profitable.

6 we establish that the classic bilateral bargaining setting of Myerson and Satterthwaite (1983) translates to our model as a single-issue mediation problem with uncertain gains. We provide a simple example that confirms the above insight by showing that it is possible to obtain a strategy-proof and efficient mechanism in that setup by adding an extra issue with certain gains.

Modeling conflicting preferences

When describing a dispute, using diametrically opposed preferences over alternatives is intuitive because it resembles a standard bargaining problem, which is modeled as a zero-sum game.[23] However, it is conceivable that many other situations, where preferences are not necessarily diametrically opposed, could also depict a dispute in which there are more than two alternatives. Consider, for example, a case where the set of available alternatives is $X = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are as follows:

θ_1	θ_2
x_1	x_3
x_2	x_5
x_3	x_4
x_4	x_2
x_5	x_1

These preferences are not diametrically opposed, but they are certainly conflicting to some extent as the agents cannot agree on their best alternative. Notice, however, that alternatives x_4 and x_5 are (Pareto) dominated by x_3, and so, if selecting an efficient outcome by the mediation protocol is desired, then the presence of these two alternatives is irrelevant for the problem. Thus, this particular dispute problem can be transformed into a reduced problem where the only available alternatives are x_1, x_2, and x_3 and the negotiators' preferences over these three are diametrically opposed. Proposition S.1. in the Supplementary Appendix generalizes this observation for any (discrete) set of alternatives and for any preference profile. A similar result, which we omit for brevity, holds for two-person, multi-issue disputes whenever preferences over bundles satisfy monotonicity.

3. The Main Model: Multi-Issue Mediation

There are two negotiators, $\mathrm{N} = \{1, 2\}$, in a dispute who aim to reach a resolution through a mediator. Without loss of generality, there are two issues that are important to the negotiators' welfare.[24] Let $\overline{X} = \{x_1, ..., x_m, o_X\}$ denote the finite set of potential outcomes in the **main issue**. Let $\overline{Y} = \{y_1, ..., y_n, o_Y\}$ denote the set of potential outcomes in the **second issue** where $n \geq m \geq 2$. We allow Y to be infinite, although we treat it as finite for expositional simplicity. We provide an extension of our model in the Supplementary Appendix where both issues have infinitely many alternatives. The sets $X = \overline{X} \backslash \{o_X\}$ and $Y = \overline{Y} \backslash \{o_Y\}$ are the sets of available **alternatives**. Each negotiator is entitled to an **outside option** (disagreement point) for each issue, respectively denoted by o_X and o_Y, in

[23] Börgers and Postl (2009) consider two agents who have diametrically opposed ordinal preferences in an arbitration problem with three alternatives (and no outside options). Their cardinal setting assumes that agents' utilities of the middle-ranked alternative are i.i.d. and privately observed. They show that there is no rule that truthfully elicits utilities and implements efficient outcomes.

[24] The extension to the case with more than two issues is discussed in Section 6.

case either negotiator refuses to accept an alternative that is available for that issue. We refer to negotiators as "she" and to the mediator as "he".

We allow for interdependence of the resolutions in the two issues. We say that the two issues are **joint** if whenever the outside option in X is selected, the outside option in Y is also selected. For example, when there is no mutually acceptable job title in an employment negotiation, i.e., the mediator has no choice but to recommend the outside option in the main issue, the compensation level is automatically set to zero.[25] Otherwise, we say that the two issues are **separate**. For example, in a family dispute where the issues are division of assets (main) and terms of child custody (second), despite failing to reach an agreement in the main issue, the two sides may have a mutual interest in a quick resolution in the second issue. In the Supplementary Appendix, we provide a number of actual mediation cases and discuss how they fit our model. Since whether issues are joint or separate makes little difference in our analysis, in the remainder of the paper we assume for expositional simplicity that the issues are separate. Nevertheless, we delegate the discussion of the resulting differences to footnotes.

Preferences over Outcomes: The negotiators' preferences over outcomes for each individual issue satisfy the following three conditions:

1. The negotiators' preferences over alternatives (not including the outside option) for each individual issue are diametrically opposed and public information.

2. Each negotiator's ranking of her outside option o_X (relative to the other alternatives) in issue X is her private information. Therefore, the main issue exhibits *uncertain gains from mediation*, i.e., neither the negotiators nor the mediator a priori know whether a mutually acceptable alternative exists.

3. It is public information that both negotiators rank the outside option o_Y in issue Y as their worst outcome. Therefore, the second issue exhibits *certain gains from mediation*.[26]

More formally, for any issue $Z \in \{\overline{X}, \overline{Y}\}$, let Θ_i^z denote the set of all linear orders (preference relations) of negotiator $i \in \mathrm{N}$ over issue Z and θ_i^z denote an ordinary element of the set Θ_i^z.[27] It is publicly known that $z_k \; \theta_1^z \; z_{k+1}$ and $z_{k+1} \; \theta_2^z \; z_k$ for all $k = 1, ..., |Z| - 1$. Namely, the negotiators' preferences over the alternatives for each issue are diametrically opposed (the first condition). The ranking of the outside option in issue X, o_X, is the negotiators' private information (the second condition). Finally, it is common knowledge that $y \; \theta_i^Y \; o_Y$ for all i and $y \in Y$ (the third condition). Therefore, the set of acceptable alternatives in issue X is privately known by the negotiators, and it is unknown to them whether there is a mutually acceptable alternative for that issue. However, any alternative

[25] A bilateral trading situation could also correspond to a case of joint issues. For example, in a multi-unit bargaining situation between a buyer and a seller over price (main) and quantity (second), if the two sides cannot agree on price, the quantity traded is zero.

[26] In fact, the outside option in issue Y need not the be the worst outcome for our results to go through. All that is needed is the availability of a set of alternatives that are efficient and individually rational. See the Supplementary Appendix for examples.

[27] A binary relation θ on set Z is called a **linear order** on Z if θ is complete, transitive, reflexive, and antisymmetric.

in issue Y is acceptable to both negotiators and efficient.[28] Note that there is a unique preference ordering in Θ_i^Y and m possible orderings in Θ_i^X. Without loss of generality, we ignore those types that declare all alternatives in X unacceptable. Let $\Theta_i = \Theta_i^X$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ the set of all type profiles. For the rest of the paper we use θ_i instead of θ_i^X to indicate negotiator i's preferences over the outcomes in issue X. However, whenever we need to distinguish i's preferences over issues X and Y, then we use θ_i^X and θ_i^Y, respectively.

Preferences over Bundles: A **bundle** (x, y) is a vector of outcomes, one for each issue, and the set $\overline{X} \times \overline{Y}$ denotes the set of all bundles. Let $\Re$ denote the set of all linear orders over $\overline{X} \times \overline{Y}$. Relation R is a standard element of the set $\Re$, and for any two bundles $b, b' \in \overline{X} \times \overline{Y}$, $b\ R\ b'$ means "b is at least as good as b'." Let P denote the strict counterpart of R.[29] We assume that the mediator asks each negotiator to report her type, i.e., her least-acceptable alternative in X, rather than her full-fledged preferences over all bundles. Negotiators' underlying preferences over bundles are then assumed to be compatible with the reported types and to satisfy certain conditions, which we shortly define.[30] To obtain the set of possible preferences compatible with the reported types, we invoke a preference map that satisfies certain regularity axioms.[31]

A preference (extension) map is a rule Λ that assigns to every negotiator i and type $\theta_i \in \Theta_i$ a nonempty set $\Lambda(\theta_i) \subseteq \Re$ of admissible orderings over bundles. $\bigcup_{\theta \in \Theta} \Lambda(\theta) \subseteq \Re^2$ is the domain of admissible preference profiles that is restricted by the preference map Λ.[32] The intuition is that although negotiator i knows her exact preferences over the bundles, the opponent and the mediator (or the modeler) believe that i's preferences are among those in the set $\Lambda(\theta_i)$, conditional on i's type being θ_i. In a cardinal setup, where the modeler assumes a specific utility function for each negotiator, Λ generates a unique ordering for each type θ_i. However, our setup allows more general domain specifications. For example, if the modeler only knows that the negotiators' preferences are consistent with expected utility theory or additively separable, then Λ is a multi-valued function. Our results hold for *all* "regular" preference maps, which are defined below.

For any negotiator i and type $\theta_i \in \Theta_i$, let $A(\theta_i) = \{x \in X |\ x\ \theta_i\ o_X\}$ denote i's set of **acceptable** alternatives in issue X. For any type profile $(\theta_1, \theta_2) \in \Theta$, let the set $A(\theta_1, \theta_2) = \{x \in X |\ x\ \theta_i\ o_X \text{ for all } i \in N\}$ denote the set of all **mutually acceptable** alternatives in issue X. In case we need to specify a type's acceptable alternatives, we use $\theta_i^x \in \Theta_i$ which denotes the preference relation (type) of negotiator i in which alternative

[28]Nevertheless, it is possible to find situations where the ranking of the outside option in both issues is the negotiators' private information. For that reason, the symmetric treatment of the outside option in both issues is formally investigated in Section 6.

[29]That is, $b\ P\ b'$ if and only if $b\ R\ b'$ but not $b'\ R\ b$.

[30]Alternatively, one could directly elicit negotiators' preferences over bundles. However, asking negotiators to report full preferences over $m \times n$ bundles is arguably impractical and cumbersome, which conflicts with the ease and convenience expected from the mediation process. Therefore, we adopt a simple preference-reporting language in line with our objective of detail-freeness.

[31]Using extension maps to deduce complete preferences is a common tool in social choice theory pioneered by Barberà (1977) and Kelly (1977) as a way to explore the strategy-proofness of social choice correspondences. For such an analysis to be carried out, individual preferences over sets are required. A typical approach is to infer this information from individual preferences over alternatives through certain extension axioms which assign to every ordering over alternatives a list of acceptable orderings over sets.

[32]For all $\theta = (\theta_1, \theta_2) \in \Theta$ we have $\Lambda(\theta) = \Lambda(\theta_1) \times \Lambda(\theta_2)$.

$x \in X$ is the least acceptable alternative. Namely, for any $x' \in X$, $x \; \theta_i^x \; x' \Longrightarrow o_X \; \theta_i^x \; x'$. We also define $B(\theta_i) = \{(x, y) \in X \times Y \mid x \; \theta_i \; o_X\}$ to be the set of all **acceptable bundles** for type θ_i. Let $\Lambda(\theta_i)|_B$ denote the restriction of all admissible preference relations over $X \times Y$ to a subset $B \subseteq X \times Y$.[33]

Definition 1. *A preference map Λ is* ***regular*** *if the following hold for all $i \in N$ and $\theta_i \in \Theta_i$:*

i. [Monotonicity] For any $x, x' \in \overline{X}$ and $y, y' \in \overline{Y}$ with $(x, y) \neq (x', y')$,

$$(x, y) \; P_i \; (x', y') \text{ for all } R_i \in \Lambda(\theta_i) \text{ whenever } x \; \theta_i^X \; x' \text{ and } y \; \theta_i^Y \; y'.$$

ii. [Consistency] For any $\theta_i' \in \Theta_i$ with $B(\theta_i) \subseteq B(\theta_i')$,

$$\Lambda(\theta_i')|_{B(\theta_i)} = \Lambda(\theta_i)|_{B(\theta_i)}.$$

iii. [Deal-breakers] For any $y \in Y$, $y' \in \overline{Y}$, and $x, x' \in X$ with $x \; \theta_i \; o_X \; \theta_i \; x'$,

$$(x, y) \; R_i \; (o_X, y') \; R_i \; (x', y) \text{ for all } R_i \in \Lambda(\theta_i).$$

The first condition (monotonicity) is a standard requirement. The second condition (consistency) requires that all types of the same negotiator rank the acceptable bundles in the same way. The third condition (deal-breakers) says that a bundle with an acceptable alternative in the main issue is always preferred over a bundle with the outside option, which in turn is always preferred over a bundle with an unacceptable alternative, regardless of the alternatives chosen for the second issue. In particular, unacceptable alternatives in issue X are "deal-breakers" for the negotiators, i.e., a bundle including an unacceptable alternative remain unacceptable regardless of the alternative it chooses for the second issue. For example, in an employment negotiation between a candidate and a company with multiple offices in different cities/countries, the candidate would have strict locational preferences that make some alternatives unacceptable regardless of the wage offered by the employer. Similarly, in family disputes, certain terms of visitation may be unacceptable independent of the outcome of the division of family property. We make this assumption for mathematical tractability, but otherwise, it is not needed for a possibility result. See Example 3 and the Supplementary Appendix for further discussion, where we provide a strategy-proof, efficient, and individually rational mediation rule in a domain of preferences that violates this assumption.

Direct Mechanisms with Veto Rights: Mediation would potentially be a very complicated, multistage game between the negotiators and the mediator. The mediation protocol, whatever the details may be, produces proposals for agreement that are always subject to unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and exercise her outside option.

A version of the revelation principle, which we prove in the Supplementary Appendix, guarantees that we can stipulate the following type of a direct mechanism without loss of

[33] The restriction of a binary relation over a set $X \times Y$ to a subset $B \subseteq X \times Y$ is the set of all pairs of bundles (b, b') in the relation for which b and b' are in B.

generality when representing mediation. The direct mechanism consists of two stages: an *announcement* stage and a *ratification* stage; and it is characterized by a mediation rule $f : \Theta \to \overline{X} \times \overline{Y}$. After being informed of her type, each negotiator i privately reports her type, $\widehat{\theta}_i$, to the mediator, who then proposes $f(\widehat{\theta}_1, \widehat{\theta}_2) \in \overline{X} \times \overline{Y}$. In the ratification stage, each party decides whether to accept or veto the proposed bundle. If both negotiators accept the proposed bundle, then it becomes the final outcome. If either or both negotiators veto the proposal, each party gets the outside option for both issues, i.e., (o_X, o_Y). Such two-stage mechanisms will be called *direct mechanisms with veto rights*. We seek direct mechanisms with veto rights in which truthful reporting of types at the announcement stage is a *dominant strategy equilibrium* and the mediator's proposals are never vetoed in equilibrium. It immediately follows from the definitions that such an equilibrium exists if and only if the mediation rule f is strategy-proof and individually rational.[34]

It is convenient to represent a mediation rule f by an $m \times m$ matrix $f = [f_{\ell,j}]_{(\ell,j) \in M^2}$, where $f_{\ell,j} = f(\theta_1^{x_\ell}, \theta_2^{x_j})$ and $M = \{1, ..., m\}$. The rows of this matrix correspond to possible types of negotiator 1 and the columns to possible types of negotiator 2.

$$f = \begin{array}{c|c|c|c|} & \theta_2^{x_1} & \cdots & \theta_2^{x_m} \\ \hline \theta_1^{x_1} & f_{1,1} & \cdots & f_{1,m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \theta_1^{x_m} & f_{m,1} & \cdots & f_{m,m} \\ \hline \end{array}$$

More specifically, in the matrix representation above, row (respectively, column) ℓ indicates the type of negotiator 1 (respectively, 2) that finds all alternatives $\{x_k \in X | k \leq \ell\}$ (respectively, $\{x_k \in X | k \geq \ell\}$) acceptable. For any reported pair of types $(\theta_1^{x_\ell}, \theta_2^{x_j})$, rule f chooses an outcome $f_{\ell,j} \in \overline{X} \times \overline{Y}$. It follows that there is a unique mutually acceptable alternative in issue X for the type pairs that correspond to an entry on the diagonal of the matrix, i.e., $\{f_{\ell,\ell} \mid \ell \in M\}$. Furthermore, there is no mutually acceptable alternative in issue X for the type pairs that correspond to an entry in the upper half of the matrix.[35] We next provide some notions that will be useful in the following results.

Let f_θ^Z and $f_{\ell,j}^Z$ denote the alternative that the mediation rule f offers in issue $Z \in \{X, Y\}$ when type profile is $\theta = (\theta_1^{x_\ell}, \theta_2^{x_j})$. Therefore, $f(\theta) = f_{\ell,j} = \left(f_{\ell,j}^X, f_{\ell,j}^Y\right) = (f_\theta^X, f_\theta^Y)$. A function $t : X \to Y$ is called **order-reversing** if for all $x, x' \in X$ and all $i \in N$, $x \ \theta_i^X \ x' \iff t(x') \ \theta_i^Y \ t(x)$. For any ℓ, j with $1 \leq j \leq \ell \leq m$ the nonempty subset $X_{j\ell} = \{x_k \in X | j \leq k \leq \ell\}$ of X is called a **connected** subset of X. Put differently, $X_{j\ell}$ is the set of mutually acceptable alternatives in issue X at the type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$. When $j = l$, this set is a singleton containing the only mutually acceptable alternative, i.e., $X_{jj} = \{x_j\}$. For any nonempty subset S of X and a partial order $\trianglerighteq$ on X, let $\underset{S}{\boldsymbol{max}} \trianglerighteq$ denote the maximal element in S with respect to $\trianglerighteq$.[36] Namely, if $x_S^* = \underset{S}{\boldsymbol{max}} \trianglerighteq$, then $x_S^* \trianglerighteq x$ for all $x \in S$. Note that such a maximal element is not guaranteed to exist under an arbitrary, e.g., incomplete, partial order.

[34]This result immediately follows from a short proof that is very similar to the proof of the revelation principle we provide in the Supplementary Appendix.

[35]When issues X and Y are joint, i.e., disagreement on issue X implies disagreement on issue Y, the upper half of the matrix above the diagonal is filled with (o_X, o_Y).

[36]A binary relation $\trianglerighteq$ on set X is called a *partial order* on X if $\trianglerighteq$ is transitive, reflexive, and antisymmetric.

The tuple $(X, \trianglerighteq)$ is called a **poset** (short for partially ordered set) if $\trianglerighteq$ is a partial order on X. For a poset $(X, \trianglerighteq)$, we say that an element $x \in X$ is an **upper bound of a subset** $S \subseteq X$ when $x \trianglerighteq x'$ for all $x' \in S$. The **least upper bound** of S is the upper bound of S that is less or equal to every upper bound of S. Namely, x is a least upper bound of S if $x' \trianglerighteq x$ for all upper bounds x' of S. Given a doubleton $\{x, x'\} \subseteq X$, let the join of x and x', denoted by $x \vee x'$, be the least upper bound of the doubleton. A poset $(X, \trianglerighteq)$ is called a **join semilattice** if every doubleton $\{x, x'\} \subseteq X$ has a least upper bound in X. A bundle b **Pareto dominates** the bundle b' if for all $(\theta_i, \theta_{-i}) \in \Theta$, where $b, b' \in B(\theta_i)$ for $i = 1, 2$, $b \ R_i \ b'$ for all $i \in N$ and $R_i \in \Lambda(\theta_i)$, and $b \ P_i \ b'$ for some $i \in N$ and $R_i \in \Lambda(\theta_i)$. Next we define the main properties we impose on mediation rules.

Definition 2. *The mediation rule f is* ***strategy-proof*** *if for all $i \in N$ and all $\theta_i \in \Theta_i$, $f(\theta_i, \theta_{-i}) \ R_i \ f(\theta_i', \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.*[37]

Definition 3. *The mediation rule f is* ***individually rational*** *if for all $i \in N$ and all $(\theta_i, \theta_{-i}) \in \Theta$, $f(\theta_i, \theta_{-i}) \ R_i \ (o_X, o_Y)$ for all $R_i \in \Lambda(\theta_i)$.*

Definition 4. *The mediation rule f is* ***efficient*** *if there exists no $(\theta_i, \theta_{-i}) \in \Theta$ and $(x', y') \in X \times Y$ such that $(x', y') \ R_i \ f(\theta_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$ and all $i \in N$, and for at least one i and $R_i \in \Lambda(\theta_i)$, $(x', y') \ P_i \ f(\theta_i, \theta_{-i})$.*[38]

Strategy-proofness requires truthful revelation of one's type to be her dominant strategy whatever her underlying preferences may be regardless of the type the opposite negotiator reports. Individual rationality guarantees an outcome at least as good as what each negotiator would receive were she to walk away from mediation and thus ensures that the mediator's proposals are never vetoed no matter what the underlying preferences are. Efficiency says that it should not be possible to find an alternative proposal that would make both parties better off at all possible preferences and one party strictly better off at some preference profile.

4. Main Results: Strategy-Proof Mediation

We start with a characterization of the set of strategy-proof, efficient, and individually rational mediation rules.

Theorem 1. *Suppose that the preference map Λ is regular and f is a strategy-proof, efficient, and individually rational mediation rule. Then there exists an injective and order-reversing function $t : X \to Y$, a partial order $\trianglerighteq$ on X, and an alternative $y \in Y$ such that*

$$f_{\ell,j} = \begin{cases} \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right), & \text{if } j \leq \ell \\ (o_X, y), & \text{otherwise} \end{cases}$$

where $x^*_{X_{j\ell}} = \max_{X_{j\ell}} \trianglerighteq$ is well-defined.

[37]It is worth noting that, coupled with the preference map Λ, this is a stronger incentive requirement than a standard strategy-proofness property that would be based on a full preference report of negotiators.

[38]Contrary to the improved strength of strategy-proofness, the dependence of the efficiency definition on the preference map Λ implies a weaker form of efficiency than would be the case under a standard requirement.

Theorem 1 says that any mediation rule satisfying the three properties, i.e., strategy-proofness, efficiency, and individual rationality, must always make selections from a special set of bundles when the set of mutually acceptable alternatives in issue X is nonempty. At these bundles, for each alternative in X, there is a corresponding distinct alternative in Y with which it must be paired. Interestingly, the order-reversing property implies that a more preferred alternative from issue X must be paired with a less preferred alternative from issue Y at these bundles. We interpret these bundles as representing possible "compromises" between the two issues. As such, we henceforth call a bundle $(x^*, t(x^*)) \in X \times Y$ a **logrolling bundle**. The set B^t consists of all the logrolling bundles. When $n = |Y| = |X| = m$, this set is unique, i.e., $t(x_k) = y_{m-k+1}$; otherwise, i.e., when $n > m$, there can be multiple such sets; hence multiple classes of rules.

The set B^t of logrolling bundles constitutes the "backbone" of every strategy-proof, efficient, and individually rational rule in the sense that the diagonal of any such rule must always be comprised of these bundles. The mediator has discretion over the choice of the **precedence order** $\trianglerighteq$ on X. Which logrolling bundle is selected below the diagonal depends on the chosen precedence order $\trianglerighteq$. Specifically, the mediator selects the highest precedence alternative among the set of mutually acceptable alternatives in the main issue and combines it with its corresponding alternative in the second issue. In other words, the logrolling bundles on the diagonal "propagate" in the southwestern direction following the precedence order $\trianglerighteq$. Theorem 3 provides a complementary visual characterization of these rules based on this insight.

When there is no mutually acceptable alternative in issue X, i.e., $j > \ell$, the mediation rule always chooses a designated disagreement bundle at which the outside option in X is coupled with some efficient alternative in Y. In this case, the mediation rule provides only a partial resolution to the dispute because of the severity of the disagreement on issue X.

For the rest of the paper, we call a mediation rule f a **logrolling rule** if it satisfies the properties in Theorem 1 and denote it by $f^{\trianglerighteq}$. The choice of the set of logrolling bundles together with the precedence order characterizes each mediation rule. Before giving a sketch of the proof of Theorem 1, we provide an example of these rules.

Example 1 (A logrolling rule): Suppose the main issue X consists of five alternatives, i.e., $m = 5$, and the second issue Y has at least five alternatives. Take a possible set of logrolling bundles

$$B^t = \{(x_1, t(x_1)), (x_2, t(x_2)), (x_3, t(x_3)), (x_4, t(x_4)), (x_5, t(x_5))\}$$

for some injective and order-reversing function $t : X \to Y$. Let us construct the logrolling rule $f^{\trianglerighteq}$ associated with the precedence order $\trianglerighteq$ where

$$\trianglerighteq: x_5 \trianglerighteq x_1 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3.$$

The main diagonal is filled with the members of the set of logrolling bundles, B^t, e.g., we have $f^{\trianglerighteq}_{1,1} = (x_1, t(x_1))$ in the first diagonal entry, $f^{\trianglerighteq}_{2,2} = (x_2, t(x_2))$ in the second diagonal entry, and so on. Suppose we would like to determine $f^{\trianglerighteq}_{3,1}$. The set of mutually acceptable alternatives are $X_{13} = \{x_1, x_2, x_3\}$. The highest precedence alternative in this set is x_1. Thus, $f^{\trianglerighteq}_{3,1} = (x_1, t(x_1))$. Similarly, to determine $f^{\trianglerighteq}_{4,2}$ we maximize $\trianglerighteq$ on $X_{24} = \{x_2, x_3, x_4\}$, which yields x_4. Hence, $f^{\trianglerighteq}_{4,2} = (x_4, t(x_4))$.

Alternatively, we can start from the diagonal and let the logrolling bundles spread in the southwestern direction following $\trianglerighteq$. Since alternative x_5 has higher precedence than all other bundles in B^t, the corresponding logrolling bundle claims all the entries to its southwest, which amounts to the set of all entries on the bottom row to the left of $f^{\trianglerighteq}_{5,5}$. The second-highest precedence bundle is x_1; and the corresponding logrolling bundle similarly claims all the unfilled entries to its southwest. Thus, starting from the entry $f^{\trianglerighteq}_{1,1}$ on the main diagonal, all the remaining empty entries on the first column fill up with $(x_1, t(x_1))$. Finally, whenever the negotiators have no mutually acceptable alternative in issue X, suppose the rule picks the bundle (o_X, y) for some $y \in Y$. The following matrix shows this logrolling rule.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	$(x_1, t(x_1))$	(o_X, y)	(o_X, y)	(o_X, y)	(o_X, y)
$\theta_1^{x_2}$	$(x_1, t(x_1))$	$(x_2, t(x_2))$	(o_X, y)	(o_X, y)	(o_X, y)
$\theta_1^{x_3}$	$(x_1, t(x_1))$	$(x_2, t(x_2))$	$(x_3, t(x_3))$	(o_X, y)	(o_X, y)
$\theta_1^{x_4}$	$(x_1, t(x_1))$	$(x_4, t(x_4))$	$(x_4, t(x_4))$	$(x_4, t(x_4))$	(o_X, y)
$\theta_1^{x_5}$	$(x_5, t(x_5))$	$(x_5, t(x_5))$	$(x_5, t(x_5))$	$(x_5, t(x_5))$	$(x_5, t(x_5))$

Figure 1: *A standard member of the logrolling rules family*

Sketch of the proof of Theorem 1: The proof follows four main steps: (1) establishing an injective and order-reversing map t from X to Y, and thus the set B^t; (2) proving that each entry of the lower half of the matrix f comes from the set B^t; (3) establishing the binary relation $\trianglerighteq$ over X that is transitive and antisymmetric; and (4) proving that each entry of the lower half of the matrix f is in fact the maximal element of a particular subset of B^t with respect to the partial order $\trianglerighteq$.

These four steps prove particular claims by utilizing the following core idea, which we call the **weak axiom of revealed precedence (WARP)**: If two distinct alternatives x, x' in issue X are mutually acceptable at some type profile and f suggests x (as part of a bundle) at that profile, then it cannot be the case that f suggests x' at another type profile where both x and x' are mutually acceptable. Therefore, whenever the set of mutually acceptable alternatives in issue X is nonempty, a strategy-proof, efficient and individually rational mediation rule behaves as if it is a single valued "choice rule" that satisfies the weak axiom of revealed preference (see Rubinstein 2012).

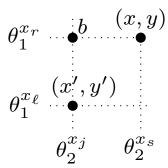


Figure 2

The intuition behind WARP is simple. Suppose it does not hold, e.g., Figure 2 indicates some entries at the lower half of the matrix form of f, where distinct alternatives x and x' of issue X are mutually acceptable by all the type profiles represented in this figure. Note that all three bundles, i.e., b, (x, y), and (x', y'), are acceptable by both types of negotiator 1 because f is individually rational and type $\theta_1^{x_\ell}$ is more accepting than type $\theta_1^{x_r}$. Strategy-proofness implies that $b \; R_1 \; (x', y')$ for all $R_1 \in \Lambda(\theta_1^{x_r})$, which is also true for all $R_1 \in \Lambda(\theta_1^{x_\ell})$ since preferences are consistent. The converse is also true, i.e., $(x', y') \; R_1 \; b$ for all $R_1 \in \Lambda(\theta_1^{x_\ell})$; since f is strategy-proof and f suggests the bundle (x', y') when player 1 announces her type as $\theta_1^{x_\ell}$. Thus, we must have $b = (x', y')$ because preferences over bundles are antisymmetric. By repeating the symmetric arguments for negotiator 2 and recalling that b and (x', y') are the same, we conclude that all these three bundles must be the same, contradicting our presumption that x and x' are distinct.

Individual rationality and efficiency of f (together with deal-breakers and monotonicity of preferences) imply that every alternative $x_k \in X$ must appear on the main diagonal of f once. We construct the injective and order-reversing function $t : X \to Y$ by setting $t(f_{k,k}^X) = f_{k,k}^Y$ for $k = 1, ..., m$. WARP implies that any entry on the second diagonal of f is equal to the main diagonal entry that is located either to its right or above. If, for example, $f_{2,1}$ and $f_{1,1}$ are the same, then negotiator 2 would profitably deviate if the function t is not order-reversing. This is true because negotiator 2 would get better alternatives in both issues by deviating to type $\theta_2^{x_2}$ rather than declaring her true type $\theta_2^{x_1}$. We denote the set of all bundles on the main diagonal by B^t (Step 1). Given that, WARP implies that each entry of the lower half of the matrix f is equal to an entry on the main diagonal of f (Step 2).

Very much like the case in rationalizable choice rules, WARP implies that f behaves as if it follows a binary relation (which we call a precedence order) over the set of alternatives in issue X such that it always picks the alternative in issue X that is revealed to be "better" than any other mutually acceptable alternative. Therefore, we construct the partial order as follows. Take any type profile $\theta = (\theta_1^{x_\ell}, \theta_2^{x_j})$ that corresponds to an entry in the lower half of the matrix f and consider the set of all acceptable alternatives in issue X at that profile, i.e., $X_{j\ell}$. We say $f_{\ell,j}^X \trianglerighteq x$ whenever $x \in X_{j\ell}$. It follows from construction that the binary relation $\trianglerighteq$ is antisymmetric and transitive and each entry of the lower half of the matrix is indeed the maximal element of a particular subset of B^t with respect to the partial order.

By individual rationality, f must choose o_X above the diagonal, i.e., when there is no mutually agreeable alternative in X. Then the fact that there must be a unique designated disagreement bundle is shown by iterating strategy-proofness along the rows and columns above the diagonal.

Full Characterization and Quid Pro Quo

Theorem 1 characterizes the necessary conditions that a strategy-proof, efficient, and individually rational mediation rule must satisfy. However, a logrolling rule is not guaranteed to be strategy-proof in general. Consider, for example, a scenario where alternatives in issue Y have little appeal for the negotiators compared to those in issue X, e.g., preferences are lexicographic over the two issues. Then there is little reason to suspect that the impossibility in the single-issue case will be overturned in the two-issue world. In such a scenario, ignoring any possible alternatives she may be assigned from issue Y, a negotiator can easily manipulate a logrolling rule by using a truncation strategy much like in the example given in Section 2. Therefore, we now search for a condition on preferences that guarantees strategy-proofness.

Since the class of logrolling rules contains the only candidates that can achieve the properties in Theorem 1, ensuring that a logrolling rule is strategy-proof automatically entails imposing a discipline on preferences regarding how negotiators rank the logrolling bundles. It then follows that the preference domain should permit the possibility that the negotiators are willing to make compromises in issue X for a more favorable treatment in issue Y. Put differently, it should be possible to find some alternatives in issue Y that are sufficiently attractive for at least one of the negotiators to reverse her ranking of some alternatives in issue X when they are bundled together.

Definition 5. *The preference map satisfies* ***quid pro quo*** *if there exists an injective and order-reversing function* $t : X \to Y$ *and a partial order* $\succeq_t$ *over* X *such that:*

i. For any distinct $x, x' \in X$*,* $x \succeq_t x'$ *if there exists* $i \in N$ *such that* $x' \; \theta_i \; x$ *and*

1. $(x, t(x)) \; R_i \; (x', t(x'))$ *for all* $R_i \in \Lambda(\theta_i)$ *and* $\theta_i \in \Theta_i$ *satisfying* $x, x' \in A(\theta_i)$*,*
2. *there is no* $y \in Y$ *with* $t(x) \; \theta_i^Y \; y \; \theta_i^Y \; t(x')$ *such that* (x', y) *Pareto dominates* $(x, t(x))$*.*

ii. The poset $(S, \succeq_t)$ *is a join semilattice for all connected* $S \subseteq X$*.*

Given a preference map Λ satisfying quid pro quo, let Π_Λ denote the set of all partial orders induced by Λ. Namely, Π_Λ is the set of all partial orders $\succeq_t$ over X such that the order-reversing function t and $\succeq_t$ satisfy Definition 5.

Quid pro quo says that negotiators should be able to "exchange favors" from the two issues to allow for reversals in preferences. Specifically, condition (i.1) says that for some pairs of acceptable alternatives x, x' in X, there must be a negotiator such that although she ranks x' above x, there is a pair of alternatives y, y' in Y such that she ranks (x, y) at least as high as (x', y') at all admissible preferences. Condition (i.2) is a type of continuity condition which requires that the function t has no "jumps" in the sense that a more efficient way to pair up the alternatives in the main issue with the alternatives in the second issue cannot be found.[39] Which alternatives in issue X allow for such reversals defines a partial order on X, and condition (ii) requires this partial order together with any connected subset of X to form a join semilattice.

Quid pro quo necessitates that the attractiveness of the alternatives in Y is sufficiently dispersed so that the negotiators are willing to trade some acceptable alternatives in issue X with other acceptable, but less desirable alternatives in X. This notion can be viewed as the nontransferable utility analogue of the *possibility of compensation* assumption in a

[39]Condition (i.2) is necessary only when $|Y| > |X|$ and is satisfied when working with standard continuous utility functions such as those in Example 2.

transferable utility model; see, e.g., Thomson (2016). It imposes a certain type of substitutability between the two issues. As such, it rules out lexicographic preferences. Although our ordinal approach does not make any explicit assumptions about (or seek to elicit) negotiators' cardinal preferences, this certainly does not preclude the possibility that the negotiators are inherently endowed with such preferences. In fact, many standard utility functions are compatible with the quid pro quo condition. This is illustrated in Example 2 for the Cobb-Douglas and quasi-linear preferences when alternatives represent quantities.

Example 2 (Quid Pro Quo Under Standard Preferences): Suppose the negotiators are in a dispute over dividing 10 units of good X and 6 units of good Y. Also suppose there are five possible ways of dividing good X where an alternative $x \in X = \{1, 3, 5, 7, 9\}$ denotes negotiator 1's share of good X.[40] Her least acceptable amount of good X is each negotiator's private information. Symmetrically, an alternative $y \in Y = [0, 6]$ denotes negotiator 1's share of good Y. Therefore, when bundle (x, y) is chosen, negotiator 1 gets (x, y) and 2 gets $(10 - x, 6 - y)$. It is commonly known that negotiator i's consumption utility from a bundle with an acceptable amount of good X and Y is given by some function U_i (and is otherwise zero when offered an unacceptable amount of good x) which may also depend on some privately known parameter(s). The private parameter in the utility function represents the uncertainty others are facing with regards to the negotiators' full fledged preferences over bundles as captured by the preference map Λ. Consider a linear order-reversing function $t(x) = \frac{11-x}{2}$ with which the mediator chooses alternatives from issue Y. This leads to one possible set of logrolling bundles where $B^t = \{(1, 5), (3, 4), (5, 3), (7, 2), (9, 1)\}$.[41]

Cobb-Douglas: Suppose $U_i(x, y) = x^{a_i} y^{b_i}$ where $\frac{a_i}{b_i} \in [\frac{7}{12}, 1]$ for $i = 1, 2$; but the exact values of a_i and b_i are only privately known. Although negotiator 1 ceteris paribus prefers higher values of x, i.e., $x' \; \theta_1 \; x$ when $x' > x$, it can be verified that $(5, 3) \; R_1 \; (7, 2) \; R_1 \; (9, 1)$. This induces the partial order $\succeq_t$ such that $5 \succeq_t 7 \succeq_t 9$. Similarly, although negotiator 2 ceteris paribus prefers lower values of x (getting more of good X), we have $(5, 3) \; R_2 \; (3, 4) \; R_2 \; (1, 5)$. This induces the partial order $\succeq_t$ such that $5 \succeq_t 3 \succeq_t 1$. A logrolling rule associated with B^t and partial order $\succeq_t$ is strategy-proof. Figure 5 in Section 5 illustrates one such rule.

Quasi-linear: Suppose $U_i(x, y) = x^{\alpha_i} + y$ where $\alpha_i \in (0, \frac{3}{5}]$ for $i = 1, 2$; but the exact value of α_i is only privately known. Although negotiator 1 ceteris paribus prefers higher values of x, i.e., $x' \; \theta_1 \; x$ when $x' > x$, she prefers $(x, t(x))$ over $(x', t(x'))$ for all $x, x' \in X$. Based solely on 1's preferences, this induces a complete order on X where $x \succeq_t^1 x'$ when $x' > x$. Similarly, negotiator 2's preferences over bundles satisfies an analogous reversal. Based solely on 2's preferences, this induces a complete order on X where $x \succeq_t^2 x'$ when $x > x'$. Consequently, any linear order on X satisfies Condition $(i1)$ in Definition 5. Indeed,

[40] While we have chosen this particular division possibility for a simple illustration with symmetric division options, it is also possible to choose $X = [0, 10]$ or some other discrete set. See the continuous analogue of our model in the Supplementary Appendix.

[41] When either issue is a continuum, there are infinitely many (possibly non-linear) order-reversing functions (i.e., infinitely many possible sets of logrolling bundles) with which the mediator can choose to form the basis of his logrolling rule. See Remark 2.

any logrolling rule with B^t and any well-defined partial order on X is strategy-proof under these preferences.

Remark 2: It is straightforward to generalize the above illustration to arbitrary well-behaved (e.g., continuous and monotonic) utility functions and arbitrary division possibilities. All that matters for the quid pro quo condition to be satisfied is how the curvature of the utility function, i.e., rate of substitution between the two issues, compares with the slope of the order-reversing (decreasing) t function that governs the set of logrolling bundles.[42] Consequently, for any given pair of differentiable and increasing utility functions, it is possible to choose an arbitrary division possibility (i.e., order-reversing function) so long as these slope comparisons have the appropriate direction and vice versa. More generally, when division possibilities are governed by a linear t function as in Example 2, quid pro quo holds when preferences are convex. Also see the continuous analogue of our model in the Supplementary Appendix allowing for non-linear division possibilities.

We are now ready to provide a full characterization result.

Theorem 2. *Given a regular preference map Λ, there exists a mediation rule f satisfying strategy-proofness, efficiency, and individual rationality if and only if Λ satisfies quid pro quo and there is a partial order $\succeq_t \in \Pi_\Lambda$ such that $f = f^{\succeq_t}$.*

Theorem 2 states that quid pro quo is both necessary and sufficient for the existence of strategy-proof, efficient, and individual rational mediation rules, and any such rule must be a logrolling rule that is generated by a precedence order $\succeq_t$ which is induced by the preference map Λ.

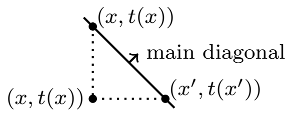
Sketch of the Proof of Theorem 2: Consider the"if" part. The mediation rule $f^{\succeq_t}$ satisfies individual rationality because it never suggests an unacceptable alternative. For efficiency, we consider a bundle $b = (x_b, t(x_b))$ that $f^{\succeq_t}$ suggests at some type profile, corresponding to the lower half of the matrix. Suppose for a contradiction that another mutually acceptable bundle $a = (x_a, t(x_a)) \in B^t$ Pareto dominates b. Because $f^{\succeq_t}$ suggests b, we must have $x_b \succeq_t x_a$. Moreover, since Λ satisfies quid pro quo, there must exist a negotiator i such that $x_a \; \theta_i \; x_b$ but $b \; R_i \; a$ for all consistent R_i, contradicting that a Pareto dominates b. If, however, $a \notin B^t$, then by condition (i.2) of Definition 5, there is no alternative $y \in Y$ that can be matched with x_a so that a Pareto dominates b. Hence, f must be efficient.

Regarding strategy-proofness of f, a profitable deviation is never possible, by the deal-breakers property, from or to a type profile in which $f^{\succeq_t}$ suggests (o_X, y). So, consider a type profile where $f^{\succeq_t}$ suggests b. Any deviation of, say, negotiator 1 to a less-accepting type to get a, which is located on the same column with b but on a lower row, is never profitable. This is true because (1) we have $x_b \succeq_t x_a$ since $f^{\succeq_t}$ suggests b when both these bundles are mutually acceptable; (2) bundle a must appear above bundle b on a lower row on the main diagonal (due to the transitivity of $\succeq_t$), i.e., $x_a \; \theta_1 \; x_b$; and thus (3) $b \; R_1 \; a$

[42]For example, if the privately known parameters are outside the given interval in the example above, quid pro quo condition can be satisfied by choosing a flatter or a steeper t function.

by quid pro quo and (1). A similar reasoning proves that negotiator 1 has no incentive to deviate to a more-accepting type. Hence, $f^{\succeq_t}$ is strategy-proof.

Consider now the "only if" part. By Theorem 1, strategy-proofness, efficiency, and individual rationality of f imply an injective and order reversing function t and a partial order $\trianglerighteq$ such that $f = f^{\trianglerighteq}$. To prove $\trianglerighteq \in \Pi_\Lambda$, and so Λ satisfies quid pro quo, we show that $\trianglerighteq$ and t satisfy Definition 5. Condition $(i.2)$ is simply implied by the efficiency of f. For condition $(i.1)$ take any x, x' with $x \trianglerighteq x'$. By the construction of $\trianglerighteq$ in the proof of Theorem 1, we know that $x \trianglerighteq x'$ implies that f must be suggesting a bundle with x at some type profile where both x and x' are mutually acceptable. Assuming, w.l.o.g., that $x\ \theta_1\ x'$, WARP requires that $f(\theta_1^{x'}, \theta_2^{x}) = (x, t(x))$ (see the figure below). Then, it is easy to verify that strategy-proofness of f implies $(x, t(x))\ R_2\ (x', t(x'))$, as required by condition $(i.1)$.



Finally, the collection of sets $X_{j\ell}$ where $1 \leq j \leq \ell \leq m$ constitutes the set of all connected subsets of X, and every doubleton $\{x, x'\} \subset X_{j\ell}$ has a least upper bound in $X_{j\ell}$, which is $x^*_{X_{j\ell}}$. Thus, $(S, \trianglerighteq)$ is a semilattice for all connected subsets of X, as required by condition (ii).

A Visual Characterization of the Class of Logrolling Rules

To provide further insight into the logrolling rules that are characterized by Theorems 1 and 2, we offer a geometric analysis of these rules. The geometric analysis will lend itself to intuitive interpretations of these rules. We first take a rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$ and introduce a couple of definitions to represent different rectangular and triangular regions of the matrix. In the following two definitions we slightly abuse notation and terminology in order to keep track of the entries contained in a rectangular/triangular region, e.g., we use $f_{\ell,j}$ to in fact refer to entry (ℓ, j) of the matrix rather than the specific bundle that rule f assigns to that entry.

Definition 6. *Consider the entry $f_{k,k}$ for some $k \in M$ and an entry that lies (weakly) to its southwest, $f_{\ell,j}$ with $1 \leq j \leq k \leq \ell \leq m$. The **rectangle** induced by $f_{k,k}$ and $f_{\ell,j}$, denoted by $\square^k_{\ell,j}$, is the set of all entries in the rectangular region of the matrix (inclusively) enveloped between rows k and ℓ and columns k and j. Namely, $\square^k_{\ell,j} = \bigcup_{\substack{j\leq s\leq k\\ k\leq t\leq \ell}} \{f_{t,s}\}$.*

Definition 7. *The **triangle** induced by an entry $f_{\ell,j}$ with $1 \leq j \leq \ell \leq m$, denoted by $\triangle_{\ell,j}$, is the set of all entries in the triangular region of the matrix that is (inclusively) enveloped by the entry $f_{\ell,j}$, row ℓ, column j, and the main diagonal. Namely, $\triangle_{\ell,j} = \bigcup_{j\leq k\leq \ell} \{f_{k,j}, f_{k,j+1}, ..., f_{k,k}\}$.*

A rectangle/triangle is merely a collection of entries of the matrix induced by rule f, i.e., sets of pairs of indexes. Note that an entry on the main diagonal is a special triangle (and also a special rectangle) that consists of a singleton entry. Furthermore, the entire main diagonal of the matrix and all the entries to its southwest constitute the largest possible triangle $\triangle_{m,1}$. Given a triangle $\triangle_{\ell,j}$, its entries that lie on the main diagonal are said to be on the *hypotenuse* of $\triangle_{\ell,j}$. A partition of the lower half of the matrix is called a *rectangular (triangular) partition* if and only if it is the union of disjoint rectangles (triangles).[43]

Theorem 3 (Visual Characterization). *The following statements are equivalent for the lower half of the mediation rule f, corresponding to type profiles in which a mutually acceptable alternative from the main issue exists;*

(*i*) *f is a logrolling rule.*

(*ii*) *The triangle $\triangle_{m,1}$ has a rectangular partition such that f assigns a unique bundle from the set of logrolling bundles B^t to each rectangle in this partition.*[44]

Part (*ii*) of Theorem 3 states that a logrolling rule f can be represented as the union of m disjoint rectangular regions. Each rectangle has a distinct corner entry on the main diagonal that contains the logrolling bundle that fills up the entire rectangle. Procedurally, these rectangles are obtained as follows. Given the precedence order, let the highest-precedence bundle on the hypotenuse of the largest triangle $\triangle_{m,1}$ fill up all the entries that are located to its southwest. This creates the first and largest rectangle $\square$, and leads to a triangular partition of $\triangle_{m,1} \backslash \square$. Next, pick any triangle from this partition and let the highest-precedence bundle on the hypotenuse of this triangle fill up all the entries that are located to its southwest. This leads to a second rectangle $\square'$ as well as a unique triangular partition of $\triangle_{m,1} \backslash \{\square, \square'\}$. The process can be iterated in this fashion until the entire triangle $\triangle_{m,1}$ is partitioned into m disjoint rectangles in m steps. Figure 3*a* provides an illustration of one such partitioning, where $b_k = (x_k, t(x_k))$ for $k = 1, ..., 9$. This process effectively traces the semilattice $(X, \trianglerighteq)$ in Figure 3*b*. Conversely, any such geometric set, namely any rectangular partition of $\triangle_{m,1}$, can be used to construct a precedence order and a corresponding logrolling rule.

Theorem 3 can be used to obtain an alternative interpretation of a logrolling rule reminiscent of how a divide-and-choose type of rule from fair division works. In particular, a logrolling rule can be thought to operate as a "shortlisting rule" in a *decentralized* fashion. One negotiator offers a short list of bundles to the other negotiator and the other negotiator chooses her favorite bundle from this list. To see this, observe that when negotiator 1 reports her type as θ^{x_ℓ}, the rule must pick a bundle on row ℓ. Equivalently, this can be viewed as negotiator 1 forming a short list consisting of all the bundles on row ℓ. Suppose that the type of negotiator 2 is θ^{x_j}. When faced with the list of bundles negotiator 1 offers her, she indeed picks $f_{\ell,j}$ since it is in fact her favorite bundle on row ℓ by strategy-proofness. If the roles of the two negotiators in the procedure were reversed, the outcome would still be the same by a symmetric argument.[45] As an example, consider the logrolling rule depicted

[43] Note that a rectangular partition consists of m disjoint rectangles. For example, $\{\square^k_{k,1}\}^m_{k=1}$ and $\{\square^k_{m,k}\}^m_{k=1}$ are two obvious rectangular partitions of $\triangle_{m,1}$. These two partitions correspond respectively to what we will later refer to as the negotiator 1- and negotiator 2-optimal rules.

[44] More formally, for any $\square$ in the partition of $\triangle_{m,1}$ and any bundles $b, b' \in \square$, $b = b'$; but for any distinct pair $\square, \square'$ in the partition of $\triangle_{m,1}$, $(x, y) \in \square$ and $(x', y') \in \square'$ implies $x \neq x'$ and $y \neq y'$.

[45] In the context of fair division, however, the outcome of divide-and-choose may be order dependent. Divide-and-choose also violates strategy-proofness.

in Figure 3*a*. Suppose negotiator 1 is of type x_3. Then we can think of her as proposing the short list $\{b_2, b_3, (o_X, y)\}$ to the other negotiator. The corresponding short lists for her types θ^{x_5} and θ^{x_7} are respectively $\{b_2, b_4, b_5, (o_X, y)\}$ and $\{b_2, b_6, b_7, (o_X, y)\}$.

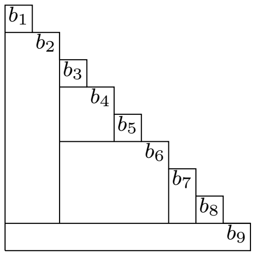


Figure 3*a*: *A rectangular partitioning of* $f^{\trianglerighteq}$ *with m=9*

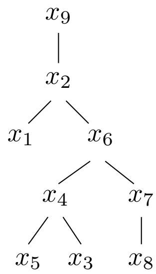


Figure 3*b*: *A semilattice* $(X, \trianglerighteq)$

Under this interpretation, a logrolling rule specifies the set of shortlisted bundles a negotiator can offer to the other party for each possible type she reports. When the proposing negotiator reports a more-accepting type, then she can add new bundles to her previous list or remove some bundles from this list. Theorem 3, however, implies that a previously removed bundle can never be added back on to the list for a more-accepting type. In our example, negotiator 1 adds bundles b_4 and b_5 to her list and removes b_3 when switching from θ^{x_3} to θ^{x_5} and adds b_6 and b_7 and removes b_4 and b_5 when switching from θ^{x_5} to θ^{x_7}. Note that once bundles b_3, b_4 or b_5 are removed, they are never added back in. Conversely, any collection of sets of shortlisted bundles (one set for each possible type of a negotiator) respecting this requirement can be shown to correspond to a logrolling rule.

5. Special members of the logrolling family

We next visit interesting members of the logrolling family. At the outset we assume that preference domain is such that all members of the family are strategy-proof, e.g., quasi-linear.[46] Three notable members of the family are worth pointing out. A **negotiator-optimal rule** represents a situation of extreme partiality to one side of the dispute and is constructed by using the precedence order implied by a negotiator's preferences over the logrolling bundles. Specifically, the negotiator 1-optimal rule takes

$$\trianglerighteq^1 : x_m \ \trianglerighteq^1 \ x_{m-1} \ \trianglerighteq^1 \ldots \trianglerighteq^1 \ x_1,$$

whereas the negotiator 2-optimal rule takes

$$\trianglerighteq^2 : x_1 \ \trianglerighteq^2 \ x_2 \ \trianglerighteq^2 \ldots \trianglerighteq^2 \ x_m.$$

In case of disagreement, i.e., when there is no mutually acceptable alternative in issue X, the corresponding designated bundle includes the favored negotiator's most-preferred alternative in issue Y. The two dual rules are shown below for the case of $m = n = 5$.

[46] This assumption renders a more meaningful comparison of the members possible. Specifically, we assume that quid pro quo is satisfied in the following strong sense. The preference map Λ admits an injective and order-reversing function $t : X \to Y$ such that for all i, $\theta_i \in \Theta_i$, $R_i \in \Lambda(\theta_i)$ and all $x, x' \in A(\theta_i)$ with $x' \ \theta_i \ x$, we have $\big(x, t(x)\big) \ R_i \ \big(x', t(x')\big)$. In this case, the negotiators' favorite logroling bundles are at the opposite corners of the diagonal.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_4}$	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_1)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 4a: *Negotiator 1-optimal rule*

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_4}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(o_X, y_5)
$\theta_1^{x_5}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 4b: *Negotiator 2-optimal rule*

A negotiator-optimal rule always chooses the corresponding negotiator's most-preferred bundle among the mutually acceptable logrolling bundles. The analogous shortlisting rule is rather simple: the favored negotiator's short list includes only two bundles, which are her favorite logrolling bundle and the designated disagreement outcome.[47] Clearly, these two polar members of the logrolling rules family are highly unattractive in practice.[48] Fortunately, there is a remarkable member of the logrolling rules family that treats negotiators symmetrically.

Impartiality entails focusing on a central element of the set of logrolling bundles as a compromise. It is then intuitive for the mediator to recommend a *median* logrolling bundle[49] when it is mutually acceptable, or seek a bundle as close to it as possible when it is not. Within the family of logrolling rules, this is achieved simply by assigning the highest precedence to a median logrolling bundle, and the next precedence to those bundles that are closest to the chosen median, and so on, and lowest precedence to the extremal logrolling bundles. Based on similar logic, when there is no mutually acceptable alternative in X, the designated bundle chosen by an impartial mediator should naturally include a median alternative in Y. This motivates the following type of rule, which we call a **constrained shortlisting (CS) rule**.

Definition 8. *Let $k \in \{\overline{k}, \underline{k}\}$ be the index of a median alternative, where $\overline{k} = \lceil \frac{m+1}{2} \rceil$ and $\underline{k} = \lfloor \frac{m+1}{2} \rfloor$. A rule is a constrained shortlisting rule, denoted $f^{CS} = [f_{\ell,j}]_{(\ell,j)\in M^2}$, if it is a logrolling rule that is associated with a precedence order $\trianglerighteq$, where $x_k \trianglerighteq x_{k-1} \trianglerighteq \ldots \trianglerighteq x_1$ and $x_k \trianglerighteq x_{k+1} \trianglerighteq \ldots \trianglerighteq x_m$, and $f^{CS}_{\ell,j} = (o_X, y_k)$ whenever $\ell < j$.*

Note that there is a unique constrained shortlisting rule when the number of alternatives is odd. When the number of alternatives is even, however, a constrained shortlisting rule prescribes one of four possible types of outcomes.[50] Figure 5 illustrates the constrained shortlisting rule for the case of $m = n = 5$.

[47] Alternatively, the nonfavored negotiator's short list includes all logrolling bundles acceptable to her together with the designated disagreement outcome.

[48] Note that despite their polarity, these rules are not dictatorial. Unlike a dictatorship, they remain individually rational and never get vetoed in equilibrium. Nevertheless, they hint at the possibility of the mediator having the power to tilt the balance in a dispute despite using a rule that meets our desiderata. As discussed earlier, such situations have been documented in the practical mediation literature, in which various field studies report biased treatment.

[49] If m is odd, there is a unique median alternative in each issue. If m is even, there are two median alternatives in each issue, in which case we assume that the mediator picks either of them.

[50] In this case, the rule depends on whether $x_{\overline{k}}$ or $x_{\underline{k}}$ has the highest precedence and whether $y_{\overline{k}}$ or $y_{\underline{k}}$ is included in the designated disagreement bundle.

When the number of alternatives is odd, the CS rule is a symmetric member of the logrolling rules family.[51] In the lower half of the matrix, it acts as a negotiator-optimal rule whenever the median alternative in issue X is not mutually acceptable and recommends the median logrolling bundle whenever the set of mutually acceptable alternatives includes the median alternative. In other words, when both negotiators find at least half of the alternatives in X acceptable, the rule chooses the median logrolling bundle; and, when one negotiator finds at least half of the alternatives acceptable while the other finds less than half of the alternatives acceptable, the rule chooses the less-accepting negotiator's favorite logrolling bundle.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 5: *Constrained shortlisting rule*

In discrete resource allocation problems where agents are endowed with ordinal preference rankings, fairness properties (together with efficiency) have often proved difficult to attain in the absence of monetary transfers or a randomization device. It is nevertheless worthwhile to investigate whether it is possible for a member of the logrolling rules family to achieve alternative fairness requirements beyond symmetry. We next formulate one such ordinal fairness notion as a normative requirement for our context.

Given the negotiators' preferences over alternatives (not including the outside option), let $r_i(z) \in M$ denote negotiator i's ranking of an acceptable alternative $z \in Z \in \{X, Y\}$. For a normalization, we re-assign ranks 1 through m to the chosen alternatives in Y, i.e., let $t(x_k) = y_{m-k+1}$ and set the ranking of the outside option to be zero as we will restrict our attention to the family of logrolling rules. Given a logrolling rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, let $f_{\ell,j} = (f^X_{\ell j}, f^Y_{\ell j}) \in \overline{X} \times \overline{Y}$ denote the bundle it proposes when the negotiators' types are $\theta_1^{x_\ell}$ and $\theta_2^{x_j}$. The *rank variance of the bundle* $f_{\ell,j}$ is defined as[52]

$$var(f_{\ell,j}) \equiv \sum_{i\in I} \left(r_i(f^X_{\ell j})\right)^2 + \left(r_i(f^Y_{\ell j})\right)^2.$$

Then, the **rank variance** of a rule f is the total sum of the rank variances of all possible outcomes of f and defined as

$$Var(f) \equiv \sum_{\ell=1}^{m} \sum_{j=1}^{m} var(f_{\ell,j}).$$

[51]When the number of alternatives is even, no logrolling rule is fully symmetric.

[52]The following formulation of rank variance assigns equal weights to both issues. One may also consider assigning different weights to different issues. Theorem 4 remains unchanged in that case due to the symmetric structure of the logrolling bundles under the normalization above.

Intuitively, the larger the differences between the two negotiators' rankings of the alternatives in a given bundle, the higher the rank variance of that bundle. For example, while never recommended by a logrolling rule, the bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Despite making one negotiator as well off as possible, they make the opposite negotiator as worse off as possible. In this sense, the larger the rank variance of a mediation rule, the more skewed it is toward extremal bundles. The next result shows that rank variance is minimized only via a CS rule within the class of logrolling rules.

Theorem 4. *A mediation rule minimizes rank variance within the class of logrolling rules if and only if it is a constrained shortlisting rule.*

The characterization of a CS rule in Theorem 4 essentially follows from the fact that the median logrolling bundle has the smallest rank variance among all logrolling bundles. Furthermore, the rank variance of a logrolling bundle grows as it gets further away from the median, e.g., the negotiator-optimal bundles have the highest rank variance. Put differently, the median bundle can be viewed as the "center of gravity" of the set of logrolling bundles. A CS rule assigns the highest precedence to the median logrolling bundle and thereby ensures that this bundle is the recommended outcome whenever it is mutually acceptable.

Remark 3: The normalization above is not without loss since the lowest rank-variance logrolling bundle may differ after the ranks in issue Y are re-assigned. Nevertheless, the rank minimizing rule in the absence of this normalization within the class of logrolling rules, which is based on the same order-reversing t function, is merely a "shifted" version of a CS rule. Indeed, in such a case the X-alternative in the lowest rank-variance logrolling bundle has the highest precedence, and as before, the precedence of the remaining alternatives is determined based on their proximity to this alternative.

6. Discussion and Extensions

In this section, we provide a general discussion of our main model in light of the results obtained so far. To this end, first, we elaborate on some of our essential modeling assumptions, discuss the role they play in driving the positive results of our paper, and offer directions in which they can be extended to cases not covered in the main exposition. Second, drawing on our findings, we consider how one can go about formulating the mediation problem in a standard Bayesian setting such as that of Myerson and Sattertwaite (1983) [henceforth MS] and offer a reconciliation of the possibility results in our setup with the impossibility result in the MS setting.

Symmetric treatment of the outside options: The asymmetric treatment of the outside options in our formulation is motivated by practical and theoretical considerations. For example, the quality of the reference letter that a former employer would be willing to write (in an employment dispute), or the terms of child custody or visitation (in a family dispute), could be considered the main issue. In this type of issues, typically every alternative is not acceptable for a negotiator. Moreover, a negotiator's ranking of the outside option is not clear to all the parties. Thus, such situations would be represented as issues with uncertain gains from mediation. In various employment, family, construction, or patent/copyright infringement disputes, litigation is the standard form of resolution in cases of disagreement when the issue is one of monetary compensation or division of property. Often in such disputes litigation, i.e., the outside option, is a very long, administratively costly, and highly inefficient process relative to other potential divisions. As such, these types of situations would be represented as issues with certain gains from mediation. (See the Supplementary Appendix for more examples.) Nonetheless, it is possible to find situations where both issues exhibit uncertain gains from mediation.

We start by exploring how our results would change if the outside option in issue Y were also treated as each negotiator's private information. In particular, we consider a relaxation of the assumption that $y\ \theta_i^Y o_Y$ for all $i \in \mathrm{N}$ and $y \in Y$. The complete formalization and the proof of the next result is deferred to Supplementary Appendix.

Proposition 1. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational, and efficient.*

Proposition 1 shows that having multiple issues alone is not sufficient to offset the tension between strategy-proofness and efficiency unless the two issues are treated asymmetrically. It is also easy to see the logical independence of the properties in this impossibility. A rule that always picks the pair (o_X, o_Y) is strategy-proof but not efficient. A dictatorship rule is strategy-proof and efficient but not individually rational. A constrained dictatorship rule, where one negotiator always chooses her favorite bundle from the set of individually rational and efficient bundles, violates strategy-proofness.

More than two issues or negotiators: As we argued earlier, the two-issue model is without loss of generality. If there are more than two issues in the mediation problem, then we can regroup these issues under two types of categories depending on whether an issue has certain or uncertain gains from mediation. In particular, let category-X be the collection of issues that exhibit uncertain gains from mediation, i.e., a negotiator's least acceptable alternative is her private information, and category-Y be the collection of issues that exhibit certain gains, i.e., it is common knowledge that the outside option is the least desirable (inefficient) outcome. Under this regrouping, each negotiator now faces a vector of alternatives for each category. The negotiators' preferences over these vectors (of alternatives) need not be diametrically opposed in general. However, as long as the negotiators' preferences are monotonic, by applying Proposition S.1 in the Supplementary Appendix, we can eliminate all inefficient vectors. This brings us back to an environment analogous to our main model, in which preferences over vectors are diametrically opposed. When there are multiple parties involved in a dispute as would be the case for community/public disputes, we can similarly regroup them to be represented by either negotiator, effectively treating them as clones of the two negotiators.

Issue-wise voting in the direct mechanism with veto rights: In our main model, we assumed that the negotiators simultaneously decide to accept or veto the proposed bundle as a whole in the ratification stage of the direct mechanism representing the mediation protocol. It does not matter whether voting in the ratification stage is simultaneous or sequential. However, it is critical that the parties vote on the proposed bundle as a whole rather than voting on each component separately. An alternative consideration would be to allow the negotiators to vote separately for each individual alternative in the proposed bundle such that unless both negotiators accept the alternative being voted on, the outside option is chosen as the final outcome in the corresponding issue. In this case, revealing one's type truthfully in the announcement stage may no longer be an optimal strategy even if the mediation rule is strategy-proof. Consider, for example, a strategy-proof negotiator-1 optimal rule when $m = n$. Suppose negotiator 1 reports a type $\theta_1^{x_\ell}$ with $\ell < m$. Suppose negotiator 2's true type is $\theta_2^{x_j}$ with $j > \ell$. When negotiator 2 reports truthfully, the rule picks the disagreement bundle (o_X, y_1) as the recommendation in the announcement stage of the direct mechanism. Both o_X and y_1 prevail in the ratification stage when voted on individually. Suppose negotiator 2 were instead to report type $\theta_2^{x_k}$ with $k \leq \ell$, in which case the rule picks the logrolling bundle $(x_\ell, y_{m-\ell+1})$ as the recommendation in the announcement stage. In the ratification stage, negotiator 2 vetoes the unacceptable alternative x_ℓ and the outcome of the mechanism is $(o_X, y_{m-\ell+1})$, i.e., 2 gains by misreporting in the announcement stage. All logrolling rules can similarly be shown to be manipulable under issue-wise voting. We conclude that there is no dominant strategy incentive compatible and efficient direct mechanism with veto rights under issue-wise voting.

The general impossibility of truthfully eliciting negotiators' private information under issue-wise voting underlines the importance of jointly resolving the two issues. In particular, bundling alternatives from different issues allows the negotiators to trade favors, which our analysis reveals to be manifested by the logrolling bundles. Consequently, to achieve dominant strategy incentives together with efficiency, it is paramount that the ratification stage only allows for voting on proposed bundles as a whole.

Reconciliation with the negative results in Bayesian settings: The influential work of MS is an important milestone in showing the difficulty of efficient trade in bargaining problems with asymmetric information. It is useful to discuss the underlying factors that are absent in the MS model, which may account for the possibility results in our model. Briefly, the mechanism design problem in MS concerns a bilateral trade between a buyer and a seller, who have private information about their valuations of a good. The mechanism has two components: the probability of trade, p, and the transfer, x, both of which are functions of the traders' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both traders receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ for the seller, where the valuations v_b, v_s are the traders' private information.

The MS model translates as a single-issue mediation problem in our setup where the transfer is the issue with uncertain gains. Indeed, the buyer (seller) prefers lower (higher) transfers but it is a priori uncertain whether a transfer leading to a mutually beneficial trade exists.[53] Efficiency in MS implies that the probability of efficient trade is generically either 0 or 1, depending on whether or not the buyer's valuation is higher than the seller's valuation. This means that probability of trade cannot be considered as a second issue

[53]Moreover, the quasi-linear utility functions in MS also satisfy the monotonicity and the quid pro quo assumptions of our model.

since we require the second issue to have at least as many alternatives as the main issue. Therefore, our result in Remark 1 is in agreement with that of MS: for both models, there is no strategy-proof, efficient, and individually rational mechanism. In fact, what is needed for a possibility is a new issue with certain gains as in the case of issue Y in our model.

To provide an illustration of the above points, in the following example we offer a simple adaptation of the MS setup in our model and demonstrate how one can overcome the impossibility by adding an extra issue:

Example 3 (Possibility in the augmented MS framework): Suppose that the seller and the buyer now negotiate not only over the terms of trade but also over the division of a unit surplus. We refer to the latter as issue Y. The valuations of the good to the buyer and the seller are v_b and v_s, respectively. We assume that each negotiator knows her valuation and believes that the opponent's valuation is distributed over $[0, 1]$ with some probability distribution. The mediator privately solicits the traders' valuations and recommends a quadruple (p, x, y_s, y_b), where p denotes the probability of trade, x is the transfer, and y_s and y_b are respectively the seller's and the buyer's share of the unit surplus. The preferences of the two traders are as follows:

$$U_b = pv_b - x + u_b(y_b) \qquad U_s = x - pv_s + u_s(y_s).$$

For simplicity, suppose that $u_b(y) = u_s(y) = y$ and each trader has only two types, $v_b, v_s \in \{0.2, 0.6\}$.

Efficiency implies that $p = 1$ if $v_b \geq v_s$, $p = 0$ if $v_s < v_b$, and $y_b + y_s = 1$. Individual rationality implies that the traders' utilities are nonnegative. Therefore, the following mechanism is strategy-proof, efficient, and individually rational:[54]

	$v_b = 0.6$		$v_b = 0.2$	
$v_s = 0.6$	$p = 1$ $x = 0.6$	$y_s = 0.3$ $y_b = 0.7$	*No trade*	$y_s = 0.5$ $y_b = 0.5$
$v_s = 0.2$	$p = 1$ $x = 0.4$	$y_s = 0.5$ $y_b = 0.5$	$p = 1$ $x = 0.2$	$y_s = 0.7$ $y_b = 0.3$

When issue Y is absent, this mechanism is not strategy-proof. In fact, it is easy to show that there is no strategy-proof, efficient, and individually rational mechanism in that case.

Two insights from this example are worth noting. First, it hints at certain parallels between the family of logrolling rules of our model and the celebrated VCG mechanisms in a combinatorial multi-item setting with transferable utility. In the transferable utility model, (cardinal) preferences over items are private information and the preferences over money are common knowledge. This is much like negotiators' preferences over issue X (where complete preference rankings including the outside option are private information) versus issue Y (where complete preference rankings including the outside option are publicly

[54]The seller of type $v_s = 0.2$ has no incentive to mimic type $v_s = 0.6$. This is true because the seller's payoff under truth-telling (which is 0.7 regardless of the buyer's type) is higher than or equal to her deviation payoffs 0.7 (if the buyer is of type $v_b = 0.6$) and 0.5 (if the buyer is of type $v_b = 0.2$). Similarly, the seller of type $v_s = 0.6$ has no incentive to mimic type $v_s = 0.2$. Her payoff under truth-telling is either 0.3 (if the buyer is type $v_b = 0.6$) or 0.5 (if the buyer is type $v_b = 0.2$). However, her deviation payoffs are 0.3 regardless of the buyer's type. Symmetric arguments apply for the buyer.

known). Example 3 shows that restoring strategy-proofness in the transferable utility setup comes to the mediator at the cost of an additional unit of surplus. This is much like a budget imbalance that may arise in a VCG mechanism, in which case the planner may be compelled to subsidize the trade. Second, Example 3 also serves to show that the deal-breakers assumption is not necessary for a possibility result in our setup. Indeed, regardless of the realization of their types, either negotiator would rather make no trade, i.e., $p = x = 0$, but get the whole surplus than make a trade via this mechanism.

7. Related Literature

We now discuss the related literature in matching/assignment, bargaining, fair division, and political economy which our paper spans.

Assignment problems have proved useful in achieving strategy-proofness and efficiency via non-dictatorial rules in a number of applications. In this context, ordinal mechanisms are well known to achieve better incentive properties than their cardinal contenders in these problems.[55] In early work, Zhou (1990) showed that no cardinal mechanism is strategy-proof, efficient, and symmetric. By contrast, ordinal mechanisms such as the random priority, are well known to attain the three properties. In two-sided one-to-one and many-to-one matching problems, however, a stable mechanism can be strategy-proof only for one side of the market (see e.g., Roth and Sotomayor 1990).[56]

In assignment/matching problems, an agent's outside option is private consumption whereas in our model it creates an externality on the other negotiator. When outside options do not exist and the issues are discrete, our setting roughly resembles a type of multi-unit assignment problem, e.g., course allocation, in which only certain assignments are feasible: there are two agents, each of whom needs to be assigned two objects, one from each of two sets A and B, where an alternative in issue X (respectively Y) represents a specific pair of objects from set A (respectively B) that must be assigned simultaneously.[57] The multi-unit assignment setting, however, provides little reason to remain optimistic for positive results. The literature contains a series of papers that show impossibility results. The main result of this literature is that the only strategy-proof and efficient rules are serial dictatorships; e.g., see Pápai (2001), Klaus and Miyagawa (2002), and Ehlers and Klaus (2003).[58] Clearly, dictatorship rules have little appeal in a dispute resolution

[55] For example, the most prominent cardinal mechanism in the context of unit-assignment problems (possibly allowing for stochastic assignments), the competitive equilibrium from equal incomes solution (Hylland and Zeckhauser 1979), is not strategy-proof. This difficulty of achieving strategy-proofness is generally attributed to the tension with efficiency since cardinal mechanisms achieve stronger welfare properties (e.g., maximization of utilitarian welfare) than ordinal mechanisms.

[56] Similar to the literature on linking decisions discussed below, a common method of circumventing these impossibilities is to resort to large market arguments by allowing for the number of participants and resources to grow. Such methods are obviously inapplicable in the context of mediation.

[57] Suppose, for example, set A contains three objects in the order of decreasing desirability, a, b, and c, where a and c are in unit supply and b has two copies. Then issue X can be viewed as consisting of the following object pairings $X = \{(a, c), (b, b), (c, a)\}$. That is, if one agent gets a, the other must get c, and b cannot be assigned together with any other object.

[58] Results continue to be negative even with stochastic mechanisms (Kojima, 2009). In the course allocation context, two notable contributions that identify non-dictatorship mechanisms are Sönmez and Ünver (2010) and Budish (2011). The former paper argues for eliciting bids from students together with ordinal preferences over courses and then using a Gale-Shapley mechanism where bids are interpreted as course priorities. However, the mechanism is strategy-proof only if the bids are treated as exogenously given. The latter paper proposes an approximately efficient mechanism that is incentive compatible in a large market.

situation. Worse still, dictatorships violate individual rationality in our model,[59] i.e., such recommendations will be vetoed in equilibrium.

Mediation has been studied in the traditional bargaining literature with incomplete information, which is primarily based on the cardinal approach discussed in the Introduction. A central question is whether private information prevents the bargainers from reaping all possible gains from trade. The mechanism design approach to this problem was pioneered by the classic paper by Myerson and Satterthwaite (1983) [MS], which shows that for a model with transferable utility, there is no ex post efficient, individually rational, and Bayesian incentive compatible mechanism when there is uncertainty about whether gains are possible.[60] As discussed in Section 6, the MS model effectively corresponds to a single-issue mediation problem with uncertain gains in our setup. On the context of mediation, specifically, there are very few papers. For a model featuring a continuum of types, Bester and Warneryd (2006) show that asymmetric information about relative strengths as an outside option in a conflict may render agreement impossible even if there is no uncertainty about the agreement being efficient. In their model, conflict shrinks the pie and agreement on a peaceful settlement is always ex post efficient. Following Bester and Warneryd (2006), Hörner et al. (2015) compare the optimal mechanisms with two types of negotiators under arbitration, mediation, and unmediated communication. They show that there is no ex post efficient and Bayesian incentive compatible mechanism: the optimal mechanism is necessarily inefficient. Compte and Jehiel (2009) consider bargaining problems where outside options are private but correlated and parties have a veto right. They show that inefficiencies are inevitable whatever the exact form of correlation, which resonates with the negative result in our benchmark model of single-issue mediation. Finally, our modeling of separate and joint outside options is in line with how issues are addressed in political bargaining; see, e.g., Chen and Eraslan (2014, 2017) for similar interpretations to us.

Obtaining a possibility result in our model hinges crucially on the availability of (at least) a second issue. Linking multiple decisions/issues to overcome welfare and incentive constraints has been a useful tool in many economic applications such as bundling of goods by a monopolist (e.g., McAfee et al. 1989), agency problems (e.g., Maskin and Tirole 1990), and logrolling in voting (e.g., Wilson 1969). A common insight in these approaches is based on applying a law of large numbers theorem to ensure that truthtelling incentives are restored in a sufficiently large market. In this vein, Jackson and Sonnenschein (2007) show that by linking different issues in many situations, including the bilateral bargaining setting of MS, it is possible to achieve outcomes that are approximately efficient in an approximately incentive compatible way as the number of issues goes to infinity. In contrast with these approaches, we establish efficiency in dominant strategies with only two issues in an application where the number of potential issues is inherently limited.

A dispute resolution problem can also be interpreted as a type of fair division problem involving indivisible items. The focal rule within the characterized family of logrolling rules, the constrained shortlisting rule, allows one negotiator to effectively reduce the set

[59] A constrained dictatorship where one negotiator maximizes her welfare among the set of mutually acceptable outcomes would satisfy individual rationality, but such a rule is easily manipulable.

[60] The MS impossibility crucially depends on types being independent. Subsequently, it was shown that efficient trade may be possible when types are correlated (e.g., Gresik 1991 and McAfee and Reny 1992).

of possible outcomes to a short list, from which the other negotiator makes her favorite selection. In that sense, the constrained shortlisting rule is reminiscent of the well-known biblical rule of divide-and-choose, which has been extensively studied in fair cake-cutting problems. Two advantages of the constrained shortlisting relative to divide-and-choose is that it is strategy-proof and its outcome is independent of the ordering of the negotiators. More generally, the fair division literature almost exclusively focuses on fairness and efficiency issues due to inherent incompatibilities with strategy-proofness similar to those in the multi-unit assignment context; see, e.g., Brams and Taylor (1996).

In our setting, in contrast to a matching or a fair division model, mediation is an entirely voluntary process. As such, the mediator has no enforcement power and the negotiators are free to walk away to exercise their private outside options. Such lack of commitment to the mediator's recommendation causes negotiators to create negative externalities on each other. When one negotiator chooses to exercise her outside option by vetoing the proposal, the other negotiator is automatically compelled to also exercise her outside option.

Absent outside options, our model also resembles a voting setting where a voting scheme aggregates individual preferences (Black 1948). This type of voting domains allows to overcome the Gibbard-Satterthwaite impossibility, and the famous median voter theorem states that the majority-rule voting system that selects the Condorcet winner, i.e., the outcome most preferred by the median voter, is strategy-proof; see Moulin (1980) for a classic generalization of this result. The constrained shortlisting rule can be viewed as similar to a Condorcet winner in the sense that it recommends the median logrolling bundle when the median is mutually acceptable for both negotiators and the closest logrolling bundle to it when it is not. Nevertheless, this connection is superficial as our model differs in several ways from a voting framework. In these voting models, there are several voters whose bliss point (peak value) is their private information, whereas in our model there are two agents (the negotiators) whose peaks in each issue are publicly known. What is private information here consists in the two negotiators' outside options, which have no analogues in a voting model. Consequently, there is no clear way to adopt such voting schemes in our setup, as they would violate individual rationality. Moreover, the above analogy between the two types of models applies only when each issue is considered separately, since negotiators' underlying joint preferences over bundles in our two-issue model are not necessarily single-peaked.[61]

Finally, with the hope of arriving at possibility results, there is a tradition of searching for strategy-proof mechanisms in restricted economic environments that make it possible to escape Arrow-Gibbard-Satterthwaite impossibilities. Well-known examples include VCG mechanisms (Vickrey 1961, Groves 1973, and Clarke 1971) for public goods and private assignment with transfers; the uniform rule (Sprumont 1991) for the distribution of a divisible private good under single-peaked preferences; generalized median-voters (Moulin 1980); proportional-budget exchange rules (Barberà and Jackson 1995) that allow for trading from a finite number of prespecified proportions (budget sets); deferred acceptance

[61] An alternative view could be based on a multi-issue voting setting. However, in multidimensional voting models where people vote on several issues, a main conclusion is that strategy-proofness effectively requires each dimension to be treated separately in the sense that each dimension should admit its own generalized median voter schemes. Our strategy-proofness result, by contrast, depends critically on having more than one dimension and relies heavily on leveraging the exchangeability between the two issues.

(Gale and Shapley 1962) and top trading cycles (Shapley and Scarf 1974, Abdülkadiroglu and Sönmez 2003) and hierarchical exchange and brokerage (Pápai 2001, Pycia and Ünver 2017). We also add to this literature by introducing and characterizing an entirely new class of strategy-proof and efficient rules.

8. Conclusion

Mediation is a preferred alternative dispute resolution method thanks to the cost-effectiveness, speed, and convenience it affords to all parties involved. The need for structured and rigorous mediation protocols in practice has often been stressed by researchers and practitioners alike. Online dispute resolution platforms are often based on automation and rely on mechanized negotiation protocols. Taking a foundational market design approach to this problem, we sought systematic mechanisms for delivering consistent, transparent, and objective recommendations. We considered rules that have a simple preference reporting language: negotiators only report their least acceptable alternatives in the main issue. It turns out that complementing the main issue with a second issue—a piece of advice often voiced by pioneers in the field—is key to achieving strategy-proof, efficient, and individually rational rules. Any such rule belongs to the family of logrolling rules, which require that the mediator's recommendation must always be a logrolling bundle when a mutual agreement is feasible, i.e., a bundle that complements a more preferred alternative in one issue with a less preferred alternative from the other. A sufficient and necessary condition for strategy-proofness is the quid pro quo property of preferences that necessitates the alternatives in the second issue to be interesting enough relative to those in the main issue. The constrained shortlisting rule is the central member within the characterized class and aims to make recommendations as close to the median logrolling bundle as possible.

Our approach marries the two distinct literatures of bargaining and matching. The former literature emphasizes the role of private information and outside options in mechanism design with transferable utility. The latter literature offers blueprints for designing robust protocols in assignment problems that often arise in practice. The multiple-assignment nature of the problem at hand in our study, however, is less than encouraging in light of the abundance of negative results in that literature. Our analysis confirms these challenges in that possibility results in our framework are also elusive unless the outside options in the two issues are treated asymmetrically. We argued that ordinal mechanisms coupled with strategy-proofness can help obtain detail-free and genuinely simple protocols for mediating disputes. Notwithstanding our emphasis on ordinality, the framework developed in this paper can accommodate both transferable and nontransferable utility settings.

APPENDIX

Proof of Theorem 1: Suppose that Λ is regular and the mediation rule f is strategy-proof, efficient, and individually rational.

The case where $\ell < j$: Individual rationality and regularity imply $f^X_{\ell,j} = o_X$. Then regularity and efficiency require $f_{\ell,j} = (o_X, y)$ for some $y \in Y$. By strategy-proofness and monotonicity, we must have $f_{\ell',j} = (o_X, y)$ for all $\ell' < j$. Similarly, $f_{\ell,j'} = (o_X, y)$ for all $\ell < j'$. Fixing j (and ℓ)

and applying the same argument for all remaining rows and columns yields $f_{\ell,j} = (o_X, y)$ whenever $\ell < j$.

The case where $\ell \geq j$:

Lemma 1 (WARP). *If $x, x' \in A(\theta) \cap A(\theta') \neq \emptyset$, $x \neq x'$ and $x = f^X_\theta$, then $f^X_{\theta'} \neq x'$.*

Proof. Let $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta'_1, \theta'_2)$ be two type profiles that correspond to the lower half of the matrix form of f, i.e., $A(\theta) \cap A(\theta') \neq \emptyset$. Suppose for a contradiction that there are two distinct alternatives $x, x' \in A(\theta) \cap A(\theta')$ such that $x = f^X_\theta$ and $f^X_{\theta'} = x'$. Suppose, without loss of generality, that θ and θ' correspond to different rows and columns (if they are on the same row or column, then we can just skip this step with bundle b). Start from the type profile that corresponds the higher row; namely the profile where negotiator 1's type is more accepting. Suppose, without loss of generality, that θ' is the higher row type profile. Let $b = f(\theta_1, \theta'_2)$ be the bundle that is on the same column with θ' and on the same row with θ. All three bundles, b, $f(\theta)$, and $f(\theta')$, are acceptable by both types of negotiator 1 because f is individually rational and θ'_1 is more accepting than type θ_1. Strategy-proofness implies that $b \ R_1 \ f(\theta')$ for all $R_1 \in \Lambda(\theta_1)$. This comparison is also true for all $R_1 \in \Lambda(\theta'_1)$ since preferences are consistent. Similarly, we must have $f(\theta') \ R_1 \ b$ for all $R_1 \in \Lambda(\theta'_1)$ since f is strategy-proof. The last two comparisons imply that $b = f(\theta')$ because preferences over bundles are antisymmetric. By repeating the symmetric arguments for negotiator 2 and recalling that $b = f(\theta')$ and both x, x' are acceptable by types θ_2 and θ'_2, we conclude that these two bundles, i.e., $f(\theta)$ and $f(\theta')$ must be the same, contradicting our presumption that x and x' are distinct alternatives. □

Lemma 2 (Existence of t). *There exists an injective order-reversing function $t : X \to Y$ such that $f_{k,k} = (x_k, t(x_k))$ for every $k = 1, ..., m$.*

Proof. Row and column k correspond to the preference profile $(\theta^{x_k}_1, \theta^{x_k}_2)$ where the only mutually acceptable alternative in issue X is x_k. Therefore, for any $1 \leq k \leq m$ efficiency and individual rationality of f and regularity of preferences imply $f^X_{k,k} = x_k$ and $f^X_{k+1,k} \in \{x_k, x_{k+1}\}$ whenever $k \neq m$. Therefore, strategy-proofness of f, monotonicity of preferences and WARP imply that $f_{k+1,k} \in \{f_{k,k}, f_{k+1,k+1}\}$.

Next, we claim that $f^Y_{k+1,k+1} \ \theta^Y_1 \ f^Y_{k,k}$ for each $k = 1, ..., m-1$. If this statement is correct, then we have the desired result if we set $t(x_k) = f^Y_{k,k}$ for all k, including t being injective because θ^Y_1 is a transitive and irreflexive relation over Y. To prove the claim take any k satisfying $1 \leq k \leq m-1$. If $f_{k+1,k} = f_{k+1,k+1}$, then strategy-proofness and monotonicity of preferences of negotiator 1 require that $f^Y_{k+1,k+1} \ \theta^Y_1 \ f^Y_{k,k}$ because otherwise negotiator 1 would profitably deviate by declaring his type as $\theta^{x_k}_1$ rather than $\theta^{x_{k+1}}_1$ as she would be getting better alternative in issue X and better or the same alternative in issue Y. On the other hand, if $f_{k+1,k} = f_{k,k}$, then strategy-proofness and monotonicity of preferences of negotiator 2 require that $f^Y_{k,k} \ \theta^Y_2 \ f^Y_{k+1,k+1}$, which implies $f^Y_{k+1,k+1} \ \theta^Y_1 \ f^Y_{k,k}$ as the negotiators' preferences over the alternatives in issue X are diametrically opposed. This completes the proof. □

Therefore, a strategy-proof, efficient, and individually rational f implies an injective order-reversing function $t : X \to Y$ and a nonempty set of bundles $B^t = \{(x, t(x)) | x \in X\}$, which constitutes the set of bundles on the main (first) diagonal. For any $1 \leq j \leq \ell \leq m$ let $B^t_{j\ell} = \{(x_k, t(x_k)) \in B^t \mid j \leq k \leq \ell\}$ denote the bundles on the main diagonal between row j and ℓ.

Construction of $\trianglerighteq$: Take any type profile $\theta = (\theta^{x_\ell}_1, \theta^{x_j}_2)$ where $1 \leq j \leq \ell \leq m$. We say $f^X_{\ell,j} \trianglerighteq x$ whenever $x \in X_{j\ell}$. WARP implies that $\trianglerighteq$ is antisymmetric and reflexive. However, it is not necessarily complete.

Lemma 3. *The binary relation $\trianglerighteq$ is transitive. That is, for any triple $x, x', x'' \in X$ where $x \trianglerighteq x'$ and $x' \trianglerighteq x''$, we have $\neg x'' \trianglerighteq x$. Furthermore, for all $1 \le j < \ell \le m$, $f_{\ell,j} = \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right) \in B^t_{j\ell}$ where $x^*_{X_{j\ell}} = \max_{X_{j\ell}} \trianglerighteq$.*

Proof. Suppose for a contradiction that $f_{\ell,j} \notin B^t_{j\ell}$. By efficiency and individual rationality $f^X_{\ell,j} \in \{x_j, x_{j+1}, ..., x_\ell\} = X_{j\ell}$. Let $f^X_{\ell,j} = x_k$ for some $j \le k \le \ell$. Lemma 2 shows that $f^X_{k,k} = x_k$, and thus by WARP we must have $f^X_{s,r} = x_k$ for all $k \le s \le \ell$ and $k \le r \le j$. Consider $f_{k,j}$: By strategy-proofness of f and monotonicity of the preferences, we must have $f_{k,j} = f_{k,k}$. Similar arguments imply $f_{\ell,j} = f_{k,j}$. Hence, $f_{\ell,j} = f_{k,k} \in B^t_{j\ell}$ and $f^Y_{\ell,j} = t(f^X_{\ell,j})$.

Suppose now for a contradiction that there exists three distinct $x, x', x'' \in X$ such that $x \trianglerighteq x'$, $x' \trianglerighteq x''$, and $x'' \trianglerighteq x$. Suppose that the three distinct bundles in B^t that correspond to these three alternatives are a, b, and c, respectively. That is, $a = (x, t(x))$, $b = (x', t(x'))$ and $c = (x'', t(x''))$. Suppose, without loss of generality, that a appears above bundle b and b appears above bundle c on the main diagonal. Similar to the arguments above, strategy-proofness of f, monotonicity of preferences and WARP imply that $f_{\ell,j} = a$, $f_{k,j} = c$, and $f_{k,\ell} = b$ (see Figure 7). Therefore, f selects b at entry (k, ℓ) while both x' and x'' are mutually acceptable, but selects c at entry (k, j) while these alternatives are still mutually acceptable, contradicting with WARP.

Finally, by construction of $\trianglerighteq$ we know that $f^X_{\ell,j} = x^*_{X_{j\ell}} \trianglerighteq x$ for all $x \in X_{j\ell}$, which completes the proof. ☐

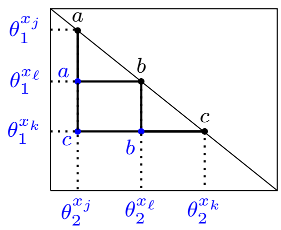


Figure 7

Proof of Theorem 2:

Proof of 'if': Suppose that a regular extension map Λ satisfies quid pro quo. By Definition 5 there exists an injective order-reversing function $t : X \to Y$ and a partial order $\succeq_t$ over X such that $\succeq_t \in \Pi_\Lambda$. Define the mediation rule $f^{\succeq_t}$ by using the partial order $\succeq_t$ as follows:

$$f^{\succeq_t}_{\ell,j} = \begin{cases} \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right), & \text{if } j \le \ell \\ (o_X, y), & \text{otherwise} \end{cases}$$

where $x^*_{X_{j\ell}} = \max_{X_{j\ell}} \succeq_t$ and $y \in Y$.

First note that $X_{j\ell}$ is a connected subset of X for all $1 \le j \le \ell \le m$ and $\succeq_t$ is a semilattice for all connected subsets of X. Thus, $\max_{X_{j\ell}} \succeq_t$ uniquely exists. Then, we need to prove that $f^{\succeq_t}$ is individually rational, efficient, and strategy-proof. It is relatively easy to verify that the rule $f^{\succeq_t}$ is individually rational: it never suggests an alternative for an issue that is worse than the outside option of that issue, and thus, it is individually rational by the regularity of preferences.

To show efficiency, first consider the type profile where both negotiators deem all alternatives acceptable in issue X, i.e., $(\theta_1^{x_m}, \theta_2^{x_1})$. Let $f^{\succeq_t}$ propose a bundle $b = (x_b, t(x_b))$ from the set $B^t = \{(x, t(x)) \mid x \in X\}$ at that profile. If instead the negotiators receive another bundle from $B^t \backslash \{b\}$ at that profile, then one of the negotiators would certainly get worse off. Suppose for a contradiction that there is another bundle $(x_a, t(x_a)) = a \in B^t \backslash \{b\}$ that Pareto dominates b. Because $f^{\succeq_t}$ suggests b, we must have $x_b = \max_X \succeq_t$, and so $x_b \succeq_t x_a$. Moreover, because Λ

satisfies quid pro quo there must exists a negotiator i where $x_a\ \theta_i\ x_b$ and $b\ R_i\ a$ for all admissible R_i, and the comparison is strict for some R_i as admissible preferences are consistent and bundles a and b are distinct. The last statement contradicts the presumption that a Pareto dominates b.

Again at that type profile, i.e., $(\theta_1^{x_m}, \theta_2^{x_1})$, if the negotiators had a bundle with the outside option in issue X, rather than b, then both negotiators would be worse off because of the deal-breakers assumption. Finally, if the negotiators had any other bundle, say $c = (x_c, y)$ where $y \neq t(x_c)$, which is neither from the set B^t nor a bundle with the outside option in issue X, then one of the negotiators would certainly get worse off. To prove this point suppose for a contradiction that the bundle c Pareto dominates b. Because $x_b = \underset{X}{\boldsymbol{max}}\ \succeq_t$, it must be the case that $x_b \succeq_t x_c$. Suppose, without loss of generality, that $x_c\ \theta_1\ x_b$. Since Λ satisfies quid pro quo and $x_c\ \theta_1\ x_b$, we must have $b\ R_1\ (x_c, t(x_c))$ for all consistent R_1's.

There are three exhaustive cases regarding the value of y. First consider the case where $t(x_c)\ \theta_1^Y\ y$. In this case, monotonicity implies $(x_c, t(x_c))\ R_1\ (x_c, y) = c$ for al admissible R_1's, and thus by transitivity we have $b\ R_1\ (x_c, t(x_c))\ R_1\ c$ for all admissible R_1, where the comparison is strict for some R_1 as admissible preferences are consistent and bundles b and c are distinct. The last statement contradicts the presumption that bundle c Pareto dominates b. On the other hand, if $y\ \theta_1^Y\ t(x_b)$, then by monotonicity $b = (x_b, t(x_b))\ P_2\ (x_c, t(x_b))\ P_2\ (x_c, y) = c$ for all admissible R_2, contradicting the presumption that bundle c Pareto dominates b. Finally, if $t(x_b)\ \theta_1^Y\ y\ \theta_1^Y\ t(x_c)$, then the bundle (x_c, y) cannot Pareto dominate the bundle b because Λ satisfies quid pro quo, which proves our initial claim that b is not Pareto dominated.

Thus, no other bundle makes one negotiator better off without hurting the other when both of the negotiators deem all alternatives acceptable. We can directly apply the same logic to all type profiles that the negotiators deem less alternatives acceptable. Finally, for those type profiles where there is no mutually acceptable alternative in issue X, in which case the rule suggests (o_X, y) for some $y \in Y$, any other bundle will include an alternative that is unacceptable in issue X by at least one of the negotiators who will veto the proposal. Thus, by regularity, at least one negotiator would be worse off if $f^{\succeq_t}$ had been proposing something other than (o_X, y). Hence, the rule $f^{\succeq_t}$ is efficient.

Next, we prove that the rule $f^{\succeq_t}$ is strategy-proof, but first establish some facts about the structure of this rule. If $a = f^{\succeq_t}_{\ell,j}$ and $b = f^{\succeq_t}_{r,s}$ are two bundles, i.e., bundle a appears on row ℓ and column j whereas bundle b appears on row r and column s, then we say bundle a appears above (below) bundle b whenever $\ell < r$ ($\ell > r$). Likewise, we say bundle a appears the right (left) of bundle b if $j > s$ ($j < s$).

Given the mediation rule $f^{\succeq_t}$ and a bundle a that appears on the main diagonal, i.e., $a = f^{\succeq_t}_{k,k}$ for some $k \in M$, define $V(a)$ to be the **value region of bundle** a, which is the submatrix of $[f^{\succeq_t}_{\ell,j}]_{(\ell,j)\in M^2}$ excluding all the rows lower than row k and all the columns higher than column k. Namely, $V(a) = [f^{\succeq_t}_{\ell,j}]_{(\ell,j)\in(M^k, M_k)}$ where $M^k = \{k, ..., m\}$ and $M_k = \{1, ..., k\}$. Furthermore, if bundle $b = f^{\succeq_t}_{r,r}$ appears on the main diagonal with $r \in M$ and $r > k$, then $V(a) \cap V(b) = [f^{\succeq_t}_{\ell,j}]_{(\ell,j)\in(M^r, M_k)}$ where $M^r = \{r, ..., m\}$. In the following figure, the value region of bundle a is region I and III, the value region of bundle b, $V(b)$, is region II and III, and $V(a)\cap V(b)$ is region III.

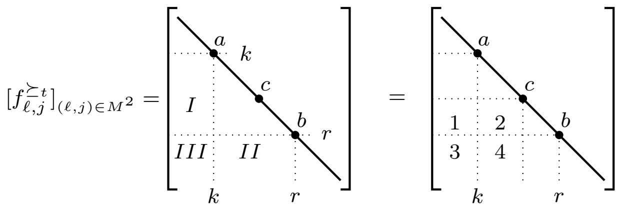
Lemma 4. *For the mediation rule $f^{\succeq_t}$ and for any two bundles $a, b \in B^t$,*

(*i*) *a never appears outside of its value region $V(a)$,*

(*ii*) *a and b both never appear in $V(a) \cap V(b)$, and*

(*iii*) *if both a and b appear on the same column (or row), where a is above b (or a is on the left of b), then on the main diagonal, bundle a appears above bundle b.*

Proof. Let $(x_a, t(x_a)) = a = f^{\succeq_t}_{k,k}$ and $(x_b, t(x_b)) = b = f^{\succeq_t}_{r,r}$ be two distinct bundles for some $k, r \in M$. The first claim follows directly from the construction of $f^{\succeq_t}$, the fact that t is one-to-one and that $x_a \notin X_{j\ell}$ for any $j \leq \ell < k$ and $k < j \leq \ell$. Transitivity of $\succeq_t$ implies the second claim but deserves a proof. Suppose first that a and b appear on the same column in region III, say column s, and a is located above bundle b on this column, namely a is on row r_a and b is on row r_b where $r \leq r_a < r_b \leq m$. Starting from column and row r, i.e., from bundle b, as we move from column r to column s along the row r, transitivity of $\succeq_t$ and the fact that the set X_{jr} is getting larger as j increases from r to s imply that the first components of the bundles on the row r, which includes the bundle $f^{\succeq_t}_{r,s}$, are either ranked higher than x_b with respect to $\succeq_t$ or equal to x_b. Now starting from column s and row r, i.e., the bundle $f^{\succeq_t}_{r,s}$, and move toward row r_a along column s. Transitivity of $\succeq_t$ and the fact that the set $X_{s\ell}$ is getting larger as ℓ increases from r to r_a imply that the first component of the bundle on row r_a and column s, i.e., the bundle a, is ranked higher than x_b with respect to $\succeq_t$. Namely, $x_a \succeq_t x_b$ must hold.



Continue iterating from where we left off with the same logic. Starting from column s and row r_a, i.e., the bundle a, as we move from row r_a to r_b along the column s, first components of the bundles, including the bundle at row r_b, i.e., b, are either ranked above x_a or equal to x_a Thus, we must have $x_b \succeq_t x_a$, contradicting the fact that $\succeq_t$ is antisymmetric and bundles a and b are distinct. If bundle b is above bundle a on column s, then we start the iteration from $f^{\succeq_t}_{k,k} = a$. Therefore, a and b cannot appear on the same column in region III. Symmetric arguments suffice to show that they cannot appear on the same row in region III either.

Therefore, suppose that a and b appear on different rows and columns. With similar arguments as above, if we start iteration from $f^{\succeq_t}_{r,r} = b$ and go left on the same row and then go down to bundle a in region III, we conclude that $x_a \succeq_t x_b$. However, when we start iteration from $f^{\succeq_t}_{k,k} = a$ and move down the same column and then go left to bundle b in region III, we conclude that $x_b \succeq_t x_a$, which yields the desired contradiction. Hence, either bundle a or b, whosever first component is ranked first with respect to $\succeq_t$, may appear in region III, but not both.

The proof of condition (iii) uses (ii). Suppose for a contradiction that a and b appear on the same column s, where b is above a (i.e., $r_b < r_a$) and a appears above b on the main diagonal. If we refer back to the previous figure, a and b can appear on the same column with $r_b < r_a$ only in region III, which contradicts what we just proved above. We can make symmetric arguments for rows as well. □

We are now ready to show that the rule $f^{\succeq_t} = [f^{\succeq_t}_{\ell,j}]_{(\ell,j)\in M^2}$ is strategy-proof. Consider, without loss of generality, deviations of negotiator 1 only. If $\ell < j$, then $A(\theta_1^{x_\ell}, \theta_2^{x_j}) = \emptyset$. Negotiator 1 may receive a different bundle by deviating to a type that is represented by a higher (numbered) row, say $\theta_1^{x_k}$ where $k > \ell$. $A(\theta_2^{x_j})$ is fixed because negotiator 2's type is fixed. Because the negotiators' preferences over issue X are diametrically opposed and $f^{\succeq_t}$ is individually rational, the alternative in issue X at type profile $(\theta_1^{x_k}, \theta_2^{x_j})$ will be unacceptable for negotiator 1's true type, $\theta_1^{x_\ell}$. Thus, by the deal-breakers property, negotiator 1 has no profitable deviation from a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell < j$.

On the other hand, if $\ell = j$, then negotiator 1 can deviate to (i) a lower row and receive (o_X, y), which is worse than $f^{\succeq_t}_{\ell,\ell} = (x_\ell, t(x_\ell))$ by deal-breakers, or (ii) a higher row and receive a bundle that suggests an unacceptable alternative in issue X. Thus, the deal-breakers property implies that negotiator 1 has no profitable deviation in that case either.

Finally, suppose that $j < \ell$. Let $c = (x_c, t(x_c)) \in B^t_{j\ell}$ denote the bundle negotiators get under truthful reporting. If negotiator 1 deviates to a row where $f^{\succeq_t}$ takes the value (o_X, y), then he clearly is worse off, by deal-breakers property. If he deviates to a lower numbered row and receives, say, bundle $a = (x_a, t(x_a)) \in B^t \backslash \{c\}$, then a appears above bundle c on the first diagonal, by the third condition of Lemma 4. In this case, it is true that $x_c \succeq_t x_a$ because $f^{\succeq_t}$ suggests c at some type profile where both a and c are acceptable and $f^{\succeq_t}$ always selects the bundle with the best first component. Moreover, by the construction of $f^{\succeq_t}$ and the fact that a appears above bundle c on the first diagonal we have $x_a \ \theta_1 \ x_c$. Since Λ satisfies quid pro quo, we must have $c \ R_1 \ a$ for all admissible R_1. Thus, there is no profitable deviation for negotiator 1 by declaring a lower numbered row and getting a instead of c.

However, if negotiator 1 declares a higher numbered row and gets a different bundle $b \in B^t \backslash \{c\}$ (see the last figure), then by the third condition of Lemma 4 bundle c must be located on the first diagonal above bundle b. As it is clearly visible in the last figure, Lemma 4 implies that negotiator 1's true preferences must give him the bundle c in region 1 or 2 and the deviation bundle b must be in region 3 or 4 because they cannot coexist in region 3 or 4. However, bundle b includes alternative x_r from issue X, which is an unacceptable alternative for all types that lie above row r, including negotiator 1's true type. Thus, by the deal-breakers property, negotiator 1 has no profitable deviation in that case either. Hence, $f^{\succeq_t}$ is strategy-proof.

***Proof of 'only if'*:** Now suppose that Λ is regular and there exists a mediation rule f that is strategy-proof, efficient, and individually rational. By Theorem 1 we know that there exists an injective order-reversing function $t : X \to Y$, a partial order $\trianglerighteq$ on X and $y \in Y$ such that

$$f = f^{\trianglerighteq}_{\ell,j} = \begin{cases} \left(x^*_{X_{j\ell}}, t\left(x^*_{X_{j\ell}}\right)\right), & \text{if } j \leq \ell \\ (o_X, y), & \text{otherwise} \end{cases}$$

where $x^*_{X_{j\ell}} = \boldsymbol{max}_{X_{j\ell}} \ \trianglerighteq$. Then we need to prove that $\trianglerighteq \in \Pi_\Lambda$, and thus Λ satisfies quid pro quo.

To prove that $\trianglerighteq$ and t satisfy the first part of Definition 5, let $x_\ell, x_j \in X$ be two distinct alternatives for some $j, \ell \in M$ and $x_\ell \trianglerighteq x_j$. Suppose, without loss of generality, that $x_j \ \theta_1 \ x_\ell$, namely $j < \ell$. Recall the construction of $\trianglerighteq$ in the proof of Theorem 1: we define $x_\ell \trianglerighteq x_j$ if $f_{\ell,j} = (x_\ell, t(x_\ell))$. Strategy-proofness of f and consistency of preferences require that $f_{\ell,j} \ R_1 \ f_{j,j}$, or equivalently $(x_\ell, t(x_\ell)) \ R_1 \ (x_j, t(x_j))$ for all admissible $R_1 \in \Lambda(\theta_1)$ and $\theta_1 \in \Theta_1$ satisfying $x_j, x_\ell \in A(\theta_1)$, as required by part (i). To prove part (ii) suppose for a contradiction that there is some $y \in Y$ with $t(x_\ell) \ \theta^Y_1 \ y \ \theta^Y_1 \ t(x_j)$ such that (x_j, y) Pareto dominates $(x_\ell, t(x_\ell)) = f_{\ell,j}$. Because both of these bundles are acceptable at the profile $(\theta^{x_\ell}_1, \theta^{x_j}_2)$, the existence of such bundle, i.e., (x_j, y), contradicts with the presumption that f is efficient.

We now prove that $\trianglerighteq$ and t satisfy the second part of Definition 5. First recall that all sets of the form $X_{j\ell}$ with $1 \leq j \leq \ell \leq m$ designate all the connected subsets of X. By Theorem 1 we already know that every doubleton $\{x, x'\} \subseteq X_{j\ell}$ has a least upper bound in $X_{j\ell}$, which is $x^*_{X_{j\ell}}$, and thus the poset $(S, \trianglerighteq)$ is a semilattice for all connected subset S of X. Hence, $\trianglerighteq \in \Pi_\Lambda$, and thus Λ satisfies quid pro quo. ■

References

[1] Abdülkadiroglu, A., and Sönmez, T. (2003). School choice: A mechanism design approach. American Economic Review, 93(3), 729-747.

[2] Ausubel, L. M., Cramton, P., and Deneckere, R. J. (2002). Bargaining with incomplete information. Handbook of game theory with economic applications, 3, 1897-1945.

[3] Barberà, S. (1977). The manipulation of social choice mechanisms that do not leave too much to chance. Econometrica 45:1573-1588.

[4] Barberà, S., and Jackson, M. O. (1995). Strategy-proof exchange. Econometrica, 63(1): 51-87.

[5] Bester, H., and Warneryd, K. (2006). Conflict and the social contract. Scandinavian Journal of Economics, 108(2), 231-249.

[6] Black, D. (1948). On the rationale of group decision-making. Journal of Political Economy, 56(1), 23-34.

[7] Bogomolnaia, A., and Moulin, H. (2001). A new solution to the random assignment problem. Journal of Economic Theory, 100(2), 295-328.

[8] Bogomolnaia, A. and Moulin, H. (2004). Random matching under dichotomous preferences. Econometrica, 72(1), pp.257-279.

[9] Börgers , T., and Postl, P. (2009). Efficient compromising. Journal of Economic Theory, 144(5), 2057-2076.

[10] Brams, S. J., and Taylor, A. D. (1996). Fair Division: From cake-cutting to dispute resolution. Cambridge University Press.

[11] Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy, 119(6), 1061-1103.

[12] Carroll, G. (2018). On mechanisms eliciting ordinal preferences. Theoretical Economics, 13(3), 1275-1318.

[13] Chen, Y. and Eraslan, H. (2014). Rhetoric in legislative bargaining with asymmetric information. Theoretical Economics, 9(2), pp.483-513.

[14] Chen, Y. and Eraslan, H. (2017). Dynamic agenda setting. American Economic Journal: Microeconomics, 9(2), pp.1-32.

[15] Clarke, E. H. (1971). Multipart pricing of public goods. Public choice, 11(1), 17-33.

[16] Compte, O., and Jehiel, P. (2009). Veto constraint in mechanism design: inefficiency with correlated types. American Economic Journal: Microeconomics, 1(1), 182-206.

[17] Crés, H., and Moulin, H. (2001). Scheduling with opting out: Improving upon random priority. Operations Research, 49(4), 565-577.

[18] Damaska, M. (1975). Presentation of Evidence and Fact finding Precision, University of Pennsylvania Law Review, 1083-1106.

[19] Ehlers, L., and Klaus, B. (2003). Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Social Choice and Welfare, 21(2), 265-280.

[20] Ergin, H., Ünver, M.U., and Sönmez, T. (2017). Dual-Donor Organ Exchange. Econometrica, 85(5), 1645-1671.

[21] Fisher, R. U., and Ury, W. (1981). Getting to Yes: Negotiating Agreement Without Giving In. A Perigee Book/Penguin Group.

[22] Gale, D., and Shapley, L. S. (1962). College admissions and the stability of marriage. The American Mathematical Monthly, 69(1), 9-15.

[23] Gresik, T. A. (1991). Ex ante incentive efficient trading mechanisms without the private valuation restriction. Journal of Economic Theory, 55(1), 41-63.

[24] Groves, T. (1973). Incentives in teams. Econometrica, 617-631.

[25] Hadfield, G., K. (2000). The Price of Law: How the Market for Lawyers Distorts the Justice System. Michigan Law Review, 98(4), 953 - 1006.

[26] Hylland, A., and Zeckhauser, R. (1979). The efficient allocation of individuals to positions. Journal of Political Economy, 87(2), 293-314.

[27] Hörner, J., Morelli, M., and Squintani, F. (2015). Mediation and peace. The Review of Economic Studies, 82(4), 1483-1501.

[28] Jackson, M.O. and Sonnenschein H.F. (2007): Overcoming Incentive Constraints by Linking Decisions. Econometrica, 75 (1), 241 - 258.

[29] Kagel, J. H., and Roth, A. E. (2016). The handbook of experimental economics (Vol. 2). Princeton University Press.

[30] Kelly, J.S. (1977). Strategy-proofness and social choice functions without single-valuedness. Econometrica 45: 439-446.

[31] Klaus, B., and Miyagawa, E. (2002). Strategy-proofness, solidarity, and consistency for multiple assignment problems. International Journal of Game Theory, 30(3), 421-435.

[32] Kojima, F., (2009). Random assignment of multiple indivisible objects. Mathematical Social Sciences, 57(1), pp.134-142.

[33] LaFree, G., and Rack, C. (1996). The Effects of Participants' Ethnicity and Gender on Monetary Outcomes in Mediated and Adjudicated Civil Cases, Law and Society Review, 30 (4): 767-798.

[34] Malhotra, D., and Bazerman, M. H. (2008). Negotiation genius: How to overcome obstacles and achieve brilliant results at the bargaining table and beyond. Bantam.

[35] Maskin, E., and Tirole, J. (1990). The principal-agent relationship with an informed principal: The case of private values. Econometrica, 379-409.

[36] McAfee, R. P., and Reny, P. J. (1992). Correlated information and mechanism design. Econometrica, 395-421.

[37] McAfee, R. P., McMillan, J., and Whinston, M. D. (1989). Multiproduct monopoly, commodity bundling, and correlation of values. The Quarterly Journal of Economics, 104(2), 371-383.

[38] Moulin, H. (1980). On strategy-proofness and single peakedness. Public Choice, 35(4), 437-455.

[39] Myerson, R. B., and Satterthwaite, M. A. (1983). Efficient mechanisms for bilateral trading. Journal of Economic Theory, 29(2), 265-281.

[40] Pápai, S. (2001). Strategyproof and nonbossy multiple assignments. Journal of Public Economic Theory, 3(3), 257-271.

[41] Pycia, M. (2014). The cost of ordinality, working paper.

[42] Pycia, M., and Ünver, M. U. (2017). Incentive compatible allocation and exchange of discrete resources. Theoretical Economics, 12(1), 287-329.

[43] Roth, A. and Sotomayor, M. (1990). Two-sided Matching: A Study in Game-theoretic Modeling and Analysis. Econometric Society Monographs, Cambridge University Press.

[44] Roth, A.E., Sönmez, T. and Ünver, M.U. (2005). Pairwise kidney exchange. Journal of Economic theory, 125(2), pp.151-188.

[45] Rubinstein, A. (2012). Lecture notes in microeconomic theory: the economic agent. Princeton University Press.

[46] Shapley, L., and Scarf, H. (1974). On cores and indivisibility. Journal of Mathematical Economics, 1(1), 23-37.

[47] Shapley, L. S., and Shubik, M. (1971). The assignment game I: The core. International Journal of Game Theory, 1(1), 111-130.

[48] Sönmez, T. and Ünver, M.U., (2010). Course bidding at business schools. International Economic Review, 51(1), pp.99-123.

[49] Sprumont, Y. (1991). The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica, 509-519.

[50] Szpilrajn, E. (1930). Sur l'extension de l'ordre partiel. Fundamenta Mathematicae, 16(1), 386-389.

[51] Thomson, W. (2016). Fair allocation, in The Oxford Handbook of Well-Being and Public Policy, ed. by M. D. Adler and M. Fleurbaey, Oxford University Press.

[52] Tyler, T. R., and Huo, Y. J. (2002). Russell Sage Foundation series on trust. Trust in the law: Encouraging public cooperation with the police and courts. New York, NY, US: Russell Sage Foundation.

[53] Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. The Journal of Finance, 16(1), 8-37.

[54] Zhou, L. (1990). On a conjecture by Gale about one-sided matching problems. Journal of Economic Theory, 52(1), 123-135.

[55] Wilson, R. (1969). An axiomatic model of logrolling. The American Economic Review, 59(3), 331-341.